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ON THE SMALL FORCED VIBRATIONS OF SYSTEMS WITH ONE DEGREE OF FREEDOM

By Maxime Bôcher

The ordinary treatment* of the problem of the small forced vibrations of a system with one degree of freedom produced by a periodic impressed force is open to two objections. On the one hand, the solution is obtained in the form of a trigonometric series [illegible] low], a form alysis of the but poorly ator to follow general not development. which the ave formu (12) below]. ctions on the ut, however, unnecessary. In the present paper I wish to show how the whole subject may be developed in a manner at once rigorous and extremely general. This treatment, which is thoroughly elementary, follows, in the deduction of Radaković's formulae, a method which, while not essentially different from that of Radaković, will, it is thought, be found more perspicuous. I have also considered various matters which are passed over without mention by Radaković, for instance the case in which the damping is so great that the *free* motion would not be vibratory.

CORRIGENDUM.

On page 189 of volume 9 of the Annals (July, 1908) the integral in the sixth line from the bottom should read

$$I_2 = \int_{R_\delta} d\bar{S} \int_{1-x_1-\ldots-x_{n-1}-\delta}^{1-x_1-\ldots-x_{n-1}} \phi dx_n.$$

---

* See for instance Lord Rayleigh's *Theory of Sound*, vol. 1, § 47.

† *Wiener Sitzungsberichte. Math. Naturwiss Klasse*, vol. 114, IIa (1905), p. 877.

‡ See Horn, *Gewöhnliche Differentialgleichungen* (*Sammlung Schubert*), §23.

ON THE SMALL FORCED VIBRATIONS OF SYSTEMS WITH ONE DEGREE OF FREEDOM

By Maxime Bôcher

The ordinary treatment* of the problem of the small forced vibrations of a system with one degree of freedom produced by a periodic impressed force is open to two objections. On the one hand, the solution is obtained in the form of a trigonometric series [see formulae (9) and (13) below], a form admirably adapted to certain acoustical questions, where the analysis of the vibration into its simple harmonic components is required, but poorly suited to the more fundamental purpose of enabling the investigator to follow the vibrating system in the details of its motion, since it is in general not easy to infer the nature of a function from its trigonometric development. The second objection refers to the correctness of the process by which the solution is obtained, as it is by no means clear that the trigonometric series found really represents a solution of the equation of motion.

The first of these objections was met by Radaković† who gave formulae for expressing the forced vibrations in closed form [cf. (5) and (12) below]. The second objection may also be met by imposing suitable restrictions on the function $f(x)$ which determines the impressed force.‡ It turns out, however, that for the correctness of the final result these restrictions are unnecessary. In the present paper I wish to show how the whole subject may be developed in a manner at once rigorous and extremely general. This treatment, which is thoroughly elementary, follows, in the deduction of Radaković's formulae, a method which, while not essentially different from that of Radaković, will, it is thought, be found more perspicuous. I have also considered various matters which are passed over without mention by Radaković, for instance the case in which the damping is so great that the *free* motion would not be vibratory.

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‡ See Horn, *Gewöhnliche Differentialgleichungen* (*Sammlung Schubert*), §23.

The problem of the forced vibration of systems with one degree of freedom when the impressed force is periodic is equivalent to the problem of solving the differential equation

$$\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + \beta y = f(x), \tag{1}$$

or, more particularly, of finding the periodic solutions of this equation, where a is a constant positive or zero which measures the amount of damping, β a positive constant, and $f(x)$ a real periodic function which we will assume to be finite for all real values of x and to have at most a finite number of discontinuities in any finite interval.

We begin by considering the case where

$$\beta > a^2 > 0, \tag{2}$$

so that the damping does not vanish and is not very great.* The general solution of the reduced equation (that is the equation obtained by replacing the second member of (1) by zero) is

$$k_1 e^{-ax}\cos\sqrt{\beta - a^2}\,x + k_2 e^{-ax}\sin\sqrt{\beta - a^2}\,x. \tag{3}$$

For the sake of brevity we write

$$s = \sqrt{\beta - a^2}.$$

By the formula derived from the method of variation of constants (see for instance Forsyth's *Treatise*, §66) we may write the general solution of (1) in the form

$$\phi(x) = k_1 e^{-ax}\cos sx + k_2 e^{-ax}\sin sx + \frac{1}{s}\int_0^x f(\xi)e^{-a(x-\xi)}\sin s(x-\xi)\,d\xi. \tag{4}$$

We wish now, if possible, to determine the constants of integration, k_1 and k_2, in such a way that $\phi(x)$ is periodic. It is clear that no solution of (1) can have a period which is not also a period of $f(x)$. Let us then denote

* This is the case in which the *free* motion is vibratory, cf. formula (3) below. If $\beta \leqq a^2$ we have for free motion the case of no oscillation illustrated by the dead-beat galvanometer.

by $2p$ any period of $f(x)$ and inquire whether k_1 and k_2 can be so determined that $\phi(x)$ also has the period $2p$, i. e., so that

$$\begin{aligned}
&k_1 e^{-a(x+2p)} \cos s(x+2p) - k_1 e^{-ax} \cos sx \\
&+ k_2 e^{-a(x+2p)} \sin s(x+2p) - k_2 e^{-ax} \sin sx \\
&+ \frac{1}{s}\int_0^{2p} f(\xi) e^{-a(x+2p-\xi)} \sin s(x+2p-\xi)\, d\xi \\
&+ \frac{1}{s}\int_{2p}^{2p+x} f(\xi) e^{-a(x+2p-\xi)} \sin s(x+2p-\xi)\, d\xi \\
&- \frac{1}{s}\int_0^{x} f(\xi) e^{-a(x-\xi)} \sin s(x-\xi)\, d\xi = 0.
\end{aligned}$$

The last two integrals here cancel on account of the periodicity of $f(\xi)$, and the equation may then be written in the form

$$\begin{aligned}
&\cos sx \left\{ k_1(1 - e^{-2pa} \cos 2ps) - k_2 e^{-2pa} \sin 2ps \right. \\
&\qquad \left. - \frac{1}{s}\int_0^{2p} f(\xi) e^{-a(2p-\xi)} \sin s(2p-\xi)\, d\xi \right\} \\
&+ \sin sx \left\{ k_1 e^{-2pa} \sin 2ps + k_2(1 - e^{-2pa} \cos 2ps) \right. \\
&\qquad \left. - \frac{1}{s}\int_0^{2p} f(\xi) e^{-a(2p-\xi)} \cos s(2p-\xi)\, d\xi \right\} = 0.
\end{aligned}$$

A necessary and sufficient condition that this equation be fulfilled for all real values of x is that the coefficients of $\cos sx$ and $\sin sx$ vanish. This gives us, for determining k_1 and k_2, two linear equations whose determinant is

$$\Delta = (1 - e^{-2pa})^2 + 4e^{-2pa} \sin^2 ps.$$

Since, under the conditions we have imposed, this quantity cannot vanish, being the sum of two real squares of which the first is not zero, we see that there is one and only one determination of k_1 and k_2, and we readily find

$$k_1 = \frac{1}{s\Delta}\int_0^{2p} f(\xi) e^{-a(2p-\xi)} [\sin s(2p-\xi) + e^{-2pa} \sin s\xi]\, d\xi,$$

$$k_2 = \frac{1}{s\Delta}\int_0^{2p} f(\xi) e^{-a(2p-\xi)} [\cos s(2p-\xi) - e^{-2pa} \cos s\xi]\, d\xi.$$

Substituting these values in (4), we find Radaković's formula

$$\phi(x) = \frac{1}{s\Delta}\int_0^{2p} f(\xi)e^{-a(x+2p-\xi)}\sin s(x+2p-\xi)\,d\xi - \frac{1}{s\Delta}\int_0^{2p} f(\xi)e^{-a(x+4p-\xi)}\sin s(x-\xi)\,d\xi + \frac{1}{s}\int_0^{x} f(\xi)e^{-a(x-\xi)}\sin s(x-\xi)\,d\xi. \tag{5}$$

We have thus proved

THEOREM 1. *If the impressed force has the period $2p$ and the damping is not zero but is sufficiently weak so that condition (2) is fulfilled, there exists one and only one forced vibration of period $2p$, and this vibration is given by (5).*

We can now see that there exists only one periodic forced vibration of any kind. For, if possible, let $\phi_1(x)$ and $\phi_2(x)$ represent two different periodic forced vibrations with periods $2p_1$ and $2p_2$ respectively. Let $2p$ denote the least common multiple of $2p_1$ and $2p_2$. Then both ϕ_1 and ϕ_2 have $2p$ as a period. This must therefore, as we have seen, be a period of $f(x)$, and therefore, by Theorem 1, there can be only one forced vibration with this period, so that our assumption has led us to a contradiction. We now easily infer

THEOREM 2. *Under the conditions of Theorem 1, there exists one and only one periodic forced vibration. If the impressed force is a constant, we have the degenerate case in which the function $\phi(x)$ which determines the vibration is a constant. Apart from this case, the periodic forced vibration has the same primitive period as the impressed force.*

Let us now consider the case in which

$$\beta < a^2 \tag{6}$$

but where a and β are still supposed to be both positive.

We will show that the analysis we have just used applies to this case also. It is true that in this case $s = \sqrt{\beta - a^2}$ is no longer real, and the trigonometric functions must be replaced by hyperbolic functions throughout. In particular we get

$$\Delta = (1 - e^{-2pa})^2 - 4e^{-2pa}\sinh^2 p\sqrt{a^2 - \beta},$$

and the reasoning by which we showed that Δ cannot vanish is no longer valid. The fact, however, is still true and may be established as follows.

We may write

$$\Delta = (1 - e^{-2p\alpha} - 2e^{-p\alpha} \sinh p\sqrt{\alpha^2 - \beta})(1 - e^{-2p\alpha} + 2e^{-p\alpha} \sinh p\sqrt{\alpha^2 - \beta})$$
$$= 4e^{-2p\alpha}(\sinh p\alpha - \sinh p\sqrt{\alpha^2 - \beta})(\sinh p\alpha + \sinh p\sqrt{\alpha^2 - \beta}).$$

Since none of these factors vanishes, $\Delta \neq 0$.

The remaining work which leads up to formula (5) requires no change, and formula (5) remains correct. If we wish, we may replace the imaginary quantity s in this formula by the real quantity $\sqrt{\alpha^2 - \beta}$, at the same time replacing sin by sinh. We thus get

THEOREM 3. *Theorems* 1 *and* 2 *remain true if condition* (2) *is replaced by condition* (6).

There remains the case $\beta = \alpha^2$. This case may be discussed along the same lines as the other two cases if we replace formula (3) by

$$k_1 xe^{-\alpha x} + k_2 e^{-\alpha x}.$$

Since no special difficulty and no special point of interest presents itself here, we content ourselves with merely stating the result, that Theorems 1 and 2 remain unchanged here too, provided in formula (5) the fractions

$$\frac{\sin s(x + 2p - \xi)}{s}, \qquad \frac{\sin s(x - \xi)}{s}$$

are replaced by $x + 2p - \xi$ and $x - \xi$ respectively.

We have thus carried through the discussion of all cases where there is damping.* The case $\alpha = 0$ presents some peculiarities to which we shall come back presently. We wish, however, first to consider the question of developing the function $\phi(x)$, which determines the periodic forced vibration, in a Fourier's series

$$\phi(x) = \frac{a_0}{2} + \sum_{i=1}^{\infty} \left(a_i \cos \frac{i\pi x}{p} + b_i \sin \frac{i\pi x}{p} \right). \tag{7}$$

That this development converges absolutely and uniformly for all values of x to the value $\phi(x)$ follows from the fact that $\phi(x)$ is not merely itself contin-

* In fact our discussion also covers the cases, which do not present themselves in the physical problem with which we are concerned, in which α and β may have any real values positive or negative, barring as yet the cases in which α or β vanish.

uous but has a continuous derivative.* The coefficients a_i and b_i, or as we will call them, following a suggestion of Hurwitz, the Fourier's constants of $\phi(x)$, might be determined by making use of the expression (5) for $\phi(x)$. They may be determined much more readily as follows by the method of undetermined coefficients.

By differentiating the series (7) term by term first once and then twice, we get new trigonometric series which we will call (7′) and (7″). Although we shall not attempt to make any statement as to whether these series represent the functions $\phi'(x)$ and $\phi''(x)$, or even as to whether they converge, we do know that their coefficients are the Fourier's constants of $\phi'(x)$ and $\phi''(x)$ respectively.† If, then, we multiply the terms of (7) by β, those of (7′) by $2a$ and add the two series thus obtained to (7″), we shall get a series whose coefficients are the Fourier's constants of $\phi'' + 2a\phi' + \beta\phi$, that is of $f(x)$.‡ If we denote the Fourier's constants of $f(x)$ by γ_i, δ_i, so that

$$(8) \qquad \gamma_i = \frac{1}{p}\int_0^{2p} f(\xi)\cos\frac{i\pi\xi}{p}\,d\xi, \qquad \delta_i = \frac{1}{p}\int_0^{2p} f(\xi)\sin\frac{i\pi\xi}{p}\,d\xi,$$

we are thus led to the equations

$$\gamma_i = -\frac{i^2\pi^2}{p^2}a_i + \frac{2i\pi}{p}ab_i + \beta a_i,$$

$$\delta_i = -\frac{i^2\pi^2}{p^2}b_i - \frac{2i\pi}{p}aa_i + \beta b_i.$$

These are linear equations for the determination of a_i and b_i in terms of the known quantities γ_i, δ_i. The determinant of these equations is

$$D_i = \frac{1}{p^4}\left\{(p^2\beta - i^2\pi^2)^2 + 4i^2p^2\pi^2a^2\right\}.$$

---

* Cf. for instance the writer's *Introduction to the Theory of Fourier's Series*, these ANNALS, vol. 7 (1906), p. 109, Theorem II; also published separately by Harvard University.

† This follows from the fact that ϕ and ϕ' are continuous, while ϕ'' is finite for all values of x and has only a finite number of discontinuities in any finite interval. Cf. the paper last referred to, p. 116, Theorem I.

‡ Cf. p. 87, Theorems V, VI, of the paper last cited.

This quantity does not vanish, since $a \neq 0$, $\beta \neq 0$. Thus we get for a_i and b_i the values

$$a_i = \frac{1}{p^2 D_i}\left\{(p^2\beta - i^2\pi^2)\gamma_i - 2ip\pi a\delta_i\right\},$$

$$b_i = \frac{1}{p^2 D_i}\left\{2ip\pi a\gamma_i + (p^2\beta - i^2\pi^2)\delta_i\right\}.$$

Substituting these values in the series (7), we get the Fourier's development of $\phi(x)$. This development may, by a simple trigonometric reduction, be thrown into the form

$$(9) \qquad \phi(x) = \frac{\gamma_0}{2\beta} + \sum_{i=1}^{\infty}\left\{\frac{\gamma_i}{\sqrt{D_i}}\cos\frac{i\pi}{p}(x - x_i) + \frac{\delta_i}{\sqrt{D_i}}\sin\frac{i\pi}{p}(x - x_i)\right\},$$

where the constants x_i are to be determined by the equations

$$(10) \qquad \cos\frac{i\pi}{p}x_i = \frac{p^2\beta - i^2\pi^2}{p^2\sqrt{D_i}}, \qquad \sin\frac{i\pi}{p}x_i = \frac{2ip\pi a}{p^2\sqrt{D_i}}.$$

Thus we have established

THEOREM 4. *The function which represents the periodic forced vibration when the damping does not vanish can always be developed into the absolutely and uniformly convergent Fourier's series* (9) *where* γ_i, δ_i *are the Fourier's constants* (8) *of the function* $f(x)$ *which determines the periodic impressed force. This last function may be any finite function with the period* $2p$ *which in every finite interval has at most a finite number of discontinuities, and need not be developable into a convergent Fourier's series.**

The case of an undamped forced vibration ($a = 0$) still remains to be considered. Here the determinants Δ and D_i become respectively

$$(11) \qquad \Delta = 4\sin^2 p\sqrt{\beta}, \qquad D_i = \frac{(p^2\beta - i^2\pi^2)^2}{p^4}.$$

In order to insure that these determinants be different from zero, we must assume that $p\sqrt{\beta}$ is not an integral multiple of π. In other words, we

* This result also, and its proof, applies without change when a, or β, or both are negative.

assume that the period $2p$ of the impressed force is not an integral multiple of the period $2\pi/\sqrt{\beta}$ of the *free* vibration. Under this assumption our analysis, so far as it refers to Theorems 1 and 3, requires no modification. Formula (5) now easily reduces to

$$(12) \qquad \phi(x) = \frac{1}{2\sqrt{\beta}\sin p\sqrt{\beta}}\int_0^{2p} f(\xi)\cos\sqrt{\beta}\,(x+p-\xi)\,d\xi + \frac{1}{\sqrt{\beta}}\int_0^x f(\xi)\sin\sqrt{\beta}\,(x-\xi)\,d\xi,$$

a formula due to Radaković.

The Fourier's development of $\phi(x)$ also simplifies, for we see from (10) that in this case all the quantities x_i vanish, and the series for $\phi(x)$ becomes

$$(13) \qquad \phi(x) = \frac{\gamma_0}{2\beta} + \sum_{i=1}^{\infty}\left\{\frac{p^2\gamma_i}{p^2\beta - i^2\pi^2}\cos\frac{i\pi x}{p} + \frac{p^2\delta_i}{p^2\beta - i^2\pi^2}\sin\frac{i\pi x}{p}\right\}.$$

We thus obtain from Theorems 1 and 3

THEOREM 5. *If there is no damping and the impressed force has a period $2p$ which is not an integral multiple of the primitive period $2\pi/\sqrt{\beta}$ of the free vibration, there exists one and only one forced vibration of period $2p$, and this is given by formula* (12) *and also by the series* (13).

Theorem 2 obviously does not admit of extension to this case except in the special case in which the periods of the impressed force and of the free vibration are incommensurable.

In the special case in which $2p$ is an integral multiple of $2\pi/\sqrt{\beta}$ it will be readily seen by a reference to the linear equations for determining k_1 and k_2 that equation (1) has no solutions with the period $2p$ unless

$$\int_0^{2p} f(\xi)\cos\sqrt{\beta}\xi\,d\xi = 0, \qquad \int_0^{2p} f(\xi)\sin\sqrt{\beta}\xi\,d\xi = 0;$$

but that if these conditions are fulfilled, every solution of (1) has the period $2p$.

HARVARD UNIVERSITY,
CAMBRIDGE, MASS.,
JULY, 1908.

TWO PRINCIPLES OF MAP-MAKING

By J. K. Whittemore

One of the most interesting applications of mathematics is to the problem of making a map of a part of a given surface, practically the most important surface being that of the earth considered as a sphere or as an oblate spheroid. The problem is essentially to establish a correspondence between points of the surface and points of a *plane* map, so that figures on the map shall resemble as closely as possible the corresponding figures of the surface. Since it is possible only for developable surfaces to construct a map on a constant scale,* that is such that all lengths on the surface are represented by proportional lengths on the map, it has been generally admitted by mathematicians, and to a considerable extent by geographers, since the publications on this subject of Lagrange and Gauss,† that the best maps are those which preserve the shapes of small figures, and in which, consequently, angles are unchanged. Such maps are called *conformal*.

Let us first consider the more general problem of mapping a surface S on any other surface S_1. The equations of the two surfaces may be written

$$S: \quad x = f(u, v), \qquad y = \phi(u, v), \qquad z = \psi(u, v);$$

$$S_1: \quad x_1 = f(u, v), \qquad y_1 = \phi_1(u, v), \qquad z_1 = \psi_1(u, v).$$

We may without loss of generality‡ suppose corresponding points of the two surfaces to be those obtained by giving to u and v the same values in the two sets of equations. If we denote by ds and ds_1 the "linear elements" on S and S_1, we have

$$ds^2 = dx^2 + dy^2 + dz^2 = \sum dx^2 = \sum \left(\frac{\partial x}{\partial u}\, du + \frac{\partial x}{\partial v}\, dv\right)^2$$
$$= E\, du^2 + 2F\, du\, dv + G\, dv^2,$$

---

* See p. 14, below.

† Lagrange, *Sur la construction des cartes géographiques.* Two memoirs presented in 1779 to the Academy of Berlin. *Oeuvres*, vol. 4, p. 637.

Gauss, *Allgemeine Auflösung der Aufgabe die Theile einer gegebenen Fläche auf einer anderen gegebenen Fläche so abzubilden, dass die Abbildung dem Abgebildeten in den kleinsten Theilen ähnlich wird;* presented in 1822 to the Royal Society of Sciences of Copenhagen, as an answer to a prize question. *Werke*, vol. 4, p. 189.

‡ Gauss, loc. cit., p. 194.

where $$E = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 = \sum \left(\frac{\partial x}{\partial u}\right)^2,$$

$$F = \sum \frac{\partial x}{\partial u}\frac{\partial x}{\partial v}, \quad G = \sum \left(\frac{\partial x}{\partial v}\right)^2.$$

Similarly, $$ds_1^2 = E_1\,du^2 + 2F_1\,du\,dv + G_1\,dv^2,$$

where $$E_1 = \sum \left(\frac{\partial x_1}{\partial u}\right)^2, \quad F_1 = \sum \frac{\partial x_1}{\partial u}\frac{\partial x_1}{\partial v}, \quad G_1 = \sum \left(\frac{\partial x_1}{\partial v}\right)^2.$$

It may be proved that it is both necessary and sufficient* for a conformal correspondence that, for all values of u and v

$$\frac{E}{E_1} = \frac{F}{F_1} = \frac{G}{G_1}.$$

If, in particular, $E = G$ and $F = 0$, so that

$$ds^2 = E(du^2 + dv^2), \tag{1}$$

and if we choose as S_1 the plane $x_1 = u$, $y_1 = v$, $z_1 = 0$, so that $ds_1^2 = du^2 + dv^2$, we shall have, by this correspondence, a conformal map of S on the plane S_1. It may be proved† that the linear element of any surface may by a suitable choice of the parameters, u and v, be thrown into the form (1) in an infinite number of ways, and further, if one such form is known, all others are obtained by writing

$$u' \pm iv' = f(u + iv),$$

where $f(u + iv)$ is any function of the complex variable $u + iv$ holomorphic in the region covered by the map.

Let us consider as examples the conformal mapping of a sphere of radius one and of an oblate spheroid obtained by rotating on its minor axis an ellipse of eccentricity e and semi-major axis one. If the center of the sphere be at the origin, and if ν be the longitude measured from the xz plane and β the

* Gauss, loc. cit., p. 195. Goursat, *Cours d'analyse mathématique*, vol. 2, p. 46.
† Gauss, loc. cit., p. 196. Darboux, *Théorie des surfaces*, vol. 1, p. 148.

complement of the latitude of a point of the sphere, we have

$$x = \sin\beta\cos v, \quad y = \sin\beta\sin v, \quad z = \cos\beta;$$

$$ds^2 = d\beta^2 + \sin^2\beta dv^2 = \sin^2\beta\left(\frac{d\beta^2}{\sin^2\beta} + dv^2\right).$$

Let
$$u = \int_{\pi/2}^{\beta}\frac{d\beta}{\sin\beta} = \log\tan\frac{\beta}{2}.$$

Then we have

$$\tan\frac{\beta}{2} = e^u, \quad \cos^2\frac{\beta}{2} = \frac{1}{1+e^{2u}}, \quad \sin^2\frac{\beta}{2} = \frac{e^{2u}}{1+e^{2u}},$$

$$\sin^2\beta = 4\sin^2\frac{\beta}{2}\cos^2\frac{\beta}{2} = \frac{4e^{2u}}{(1+e^{2u})^2} = \left(\frac{2}{e^u+e^{-u}}\right)^2,$$

so that, finally

$$ds^2 = \left(\frac{2}{e^u+e^{-u}}\right)^2(du^2+dv^2).$$

The point of longitude v, latitude $90^\circ - \beta$, is now represented by the point of coordinates (u, v) in the plane S_1 and the represention is a conformal map which may be used to represent any part of the sphere not including either pole. The map so constructed is the famous "Mercator's projection" in which circles of latitude and longitude are represented by sets of parallel straight lines, and in which a rhumb line, crossing all parallels of latitude at the same angle, is also represented by a straight line.

If we write

$$u' + iv' = e^{u+iv},$$

then is

$$u + iv = \log(u' + iv') = \log\rho + i\theta,$$

where ρ and θ are the absolute value and angle of the complex number $u' + iv'$. We have

$$u = \log\rho, \qquad v = \theta,$$

$$ds^2 = \left(\frac{2}{e^u+e^{-u}}\right)^2(du^2+dv^2) = \left(\frac{2}{\rho^2+1}\right)^2(d\rho^2+\rho^2d\theta^2).$$

In the u', v' plane, in which ρ and θ are polar coordinates, circles of latitude of the sphere (β constant) are represented by concentric circles (ρ constant),

and circles of longitude (r constant) by straight lines radiating from the origin (θ constant). The conformal map of the sphere so constructed is that obtained by stereographic polar projection.

To construct a map of the oblate spheroid obtained by revolving about its minor axis the ellipse

$$x^2 + \frac{y^2}{1 - e^2} - 1 = 0,$$

let us, following Lagrange,* express the coordinates of a point of the ellipse in terms of β, the complement of the geographical latitude, which is the angle made by the normal to the surface of revolution with the plane of the equator. One may easily see from the properties of the normal to an ellipse that $\operatorname{ctn}\beta = \dfrac{y}{x(1-e^2)}$, whence we find

$$x = \frac{\sin\beta}{\sqrt{1 - e^2\cos^2\beta}}, \qquad y = \frac{(1-e^2)\cos\beta}{\sqrt{1 - e^2\cos^2\beta}}.$$

If now v be the longitude of a point on the spheroid we shall have for its linear element

$$ds^2 = dx^2 + dy^2 + x^2\,dv^2$$

$$= \frac{\sin^2\beta}{1 - e^2\cos^2\beta}\left(\frac{(1-e^2)^2}{\sin^2\beta(1 - e^2\cos^2\beta)^2}\,d\beta^2 + dv^2\right).$$

Let us write

$$u = \int_{\pi/2}^{\beta} \frac{(1-e^2)\,d\beta}{\sin\beta(1 - e^2\cos^2\beta)} = \log\left\{\left(\frac{1 + e\cos\beta}{1 - e\cos\beta}\right)^{e/2}\tan\frac{\beta}{2}\right\}.$$

Then we have

$$ds^2 = \frac{\sin^2\beta}{1 - e^2\cos^2\beta}\,(du^2 + dv^2).$$

The coefficient would naturally be expressed, as in the case of the sphere, in terms of u, but such an expression cannot be obtained in any simple finite

* Lagrange, loc. cit. p. 661. Lagrange and other writers use the letter z for the complement of the geographical latitude. I have used β, since in this paper z is used for one of the coordinates.

form. It is clear that these formulas reduce to those for the sphere when $e = 0$. Lagrange remarks that if we write

$$\tan \frac{\zeta}{2} = \left(\frac{1 + e\cos\beta}{1 - e\cos\beta}\right)^{e/2} \tan \frac{\beta}{2},$$

we have $u = \log \tan \zeta/2$, and ζ is an angle obtained by making a correction, which vanishes with e, in the geographical co-latitude. We have thus a map of the oblate spheroid in which circles of latitude and longitude are, as in Mercator's projection, represented by sets of parallel straight lines. The transformation considered in that case would give us on the $u'v'$ plane a map of the surface in which these curves would be represented by concentric circles and lines radiating from the center.

Having seen that any surface may be mapped conformally on a plane in various ways, we now consider the *scale* of the map, on which is to depend the choice of the best conformal map. The scale is the ratio of corresponding lengths on the map and the surface at a point. Denoting the scale of the map by m_1, we have $m_1 = ds_1/ds$. Suppose a second conformal map constructed on the $u'v'$ plane, S', where

$$u' \pm iv' = f(u + iv).$$

On this map the scale m is evidently ds'/ds. We have

$$ds^2 = E(du^2 + dv^2), \qquad ds_1^2 = du^2 + dv^2,$$

$$ds'^2 = du'^2 + dv'^2 = E'(du^2 + dv^2).$$

Since

$$du' \pm i\,dv' = f'(u + iv)(du + i\,dv),$$

$$du' \mp i\,dv' = \bar{f}'(u - iv)(du - i\,dv),$$

where $\bar{f}$ is the function conjugate to f, and f' and $\bar{f}'$ denote the derivatives of f and $\bar{f}$, we have

$$E' = f'(u + iv)\,\bar{f}'(u - iv), \qquad m = \sqrt{\frac{E'}{E}}.$$

We have evidently

$$\log m = U - \tfrac{1}{2} \log E,$$

where

$$U = \tfrac{1}{2} \log E' = \tfrac{1}{2} \log f'(u + iv) + \tfrac{1}{2} \log \bar{f}'(u - iv)$$

and is obviously a solution of

$$\Delta U \equiv \frac{\partial^2 U}{\partial u^2} + \frac{\partial^2 U}{\partial v^2} = 0,$$

being simply the real part of the monogenic function $\log f'(u + iv)$. Moreover, a function f may be found such that U is any harmonic function, that is, a solution of $\Delta U = 0$, continuous with its partial derivatives in the region of the first map.

We may now see clearly that not every surface can be mapped on a constant scale, for, in the first place such a map must be comformal,* so we shall have

$$\Delta \log m = -\tfrac{1}{2} \Delta \log E,$$

and if m may be constant we must have $\Delta \log E = 0$, and this is not the case in either example considered. It is interesting and, for the purpose of our discussion, important to find another expression for $\Delta \log m$. Gauss obtained† an expression for the total curvature of any surface in terms of the coefficients E, F, G, and their derivatives which becomes, when $ds^2 = E(du^2 + dv^2)$,

$$K = -\frac{\Delta E}{2E^2} + \frac{1}{2E^3}\left[\left(\frac{\partial E}{\partial u}\right)^2 + \left(\frac{\partial E}{\partial v}\right)^2\right].$$

But we find

$$\Delta \log E = \frac{E\Delta E - \left[\left(\frac{\partial E}{\partial u}\right)^2 + \left(\frac{\partial E}{\partial v}\right)^2\right]}{E^2}.$$

Hence

$$\Delta \log E = -2EK \quad \text{and} \quad \Delta \log m = EK.$$

Since E is always positive, a map of a surface can have a constant scale only when $K = 0$, that is, only when the surface is developable.

The construction of a second conformal map on a plane S', has been shown to depend on the choice of a harmonic function of u and v on the map in the plane S_1. The problem which we shall now consider is the most interesting question in the theory of map-making, the choice of this harmonic

* Gauss, loc cit., p. 194.

† *Disquisitiones generales circa superficies curvas*, presented in 1827 to the Royal Academy of Göttingen. *Werke*, vol. 4, p. 232.

function U, so that the map on the plane S' may be the best possible map. We must be guided in this choice by a consideration of the variation of the scale of the map. Various criteria for the best conformal map of a region of a surface have been proposed. Of these I shall consider fully two* which, very curiously, starting from totally different points of view reach, in most cases, the same conditions for the determination of U. The first of these principles was enunciated in 1856 by Tchébychef, in a paper presented to the Imperial Academy of Sciences of St. Petersburg,† as follows: The minimum deviation of an integral of $\Delta U = 0$ from the function

$$\log \frac{2}{e^u + e^{-u}}$$

in a region bounded by a closed curve cannot occur unless

$$U - \log \frac{2}{e^u + e^{-u}}$$

is constant on the contour. It is not entirely obvious what Tchébychef meant by the "deviation," and his statement applies only to maps of a sphere. So far as I can learn he never published any proof of this remarkable theorem, though he applied the principle to the construction of a map of Russia. I believe that the theorem was first proved in 1894 by D. A. Gravé, a professor at St. Petersburg.‡ This proof I shall here present in a form so generalized that the principle may be applied to the construction of a map of a region of any surface in which the total curvature is throughout of one sign, thus justifying the application of the principle made in the following paper by Dr. G. W. Hill to a map of a region of an oblate spheroid.

Another criterion was proposed and discussed by F. Eisenlohr§ in 1870. He regards as a measure of the accuracy of a map the distortion of the geodesic lines of the surface. If we find at each point of the map the maximum value of the curvature of the representations of all geodesic lines through

* Two other conditions for determining U were proposed by H Weber in a paper published in *Crelle's Journal für die Mathematik*, vol. 67, p. 229, and cited by Eisenlohr. These conditions while apparently as reasonable as those of Tchébychef and Eisenlohr lead to no simple determination of U and are consequently of comparatively little interest.

† *Bulletin*, vol. 14, p. 257; *Oeuvres*, vol. 1, p. 233.

‡ *Association française pour l'avancement des sciences*, 23e session, Caen, 1894, p. 196.

§ *Crelle's Journal für die Mathematik*, vol. 72, p. 143. This paper and Weber's are cited by Gravé, loc. cit.

this point we may determine U so that the mean value on the map of the square of this maximum curvature shall be a minimum, and find, at least in the case when the *first* map in the plane S_1 is simply connected, that this condition requires that the scale be constant on the contour. It is clear that Eisenlohr's and Tchébychef's conditions are the same when the part of the surface mapped is simply connected and has a total curvature everywhere of the same sign. I shall here give a proof of Eisenlohr's principle, supplying some details omitted by him, and emphasizing the fact, which he does not consider, that the proof applies only when the map in the plane S_1 is a simply connected region.

We now take up the proof of Tchébychef's principle. Let the region of the map in the plane S_1 be denoted by T, and the contour of this region be C. The contour C may consist of one or more curves and is considered as belonging to the region T, which is therefore "bounded." Let U be a single valued harmonic function in T, that is to say continuous with all its partial derivatives and satisfying the equation $\Delta U = 0$. We shall need the following theorems* concerning U:

a) U can have no maximum or minimum value at an interior point of T. From (a) it follows that

b) If U is negative at all points of C it is negative at all points of T. From this fact it may be proved that

c) If U vanishes at some points of C and is negative at all others, it is negative at all interior points of T.†

d) If U is constant on C, it has the same constant value at all points of T, and consequently

e) If U and U' are two harmonic functions in T equal at all points of C, then since $U - U'$ is harmonic in T and also on C, U and U' are equal at all points of T.

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*See *e. g.* a discussion of harmonic functions in Osgood's *Functionentheorie*, vol. 1, p. 543.

† Theorem (c), on which the following proof depends, is, I think, not generally stated in treatises on harmonic functions. Its truth may be easily seen as follows: under the conditions of (c) U cannot be positive at an interior point of T: if at an interior point a, U vanishes, we should have by the mean value theorem (Osgood, loc. cit., p. 544) $\int_0^{2\pi\rho} U ds = 0$, where the integral is taken around the circumference of a circle of radius ρ, wholly within T, and having its center at a. Since U is not positive at any point of the circle, it is zero at all points of the circumference, and consequently, since it is continuous, at all points of C, contrary to the hypothesis.

Gravé states without proof the following theorem: If $\phi(u)$ is, with its first and second derivatives, $\phi'(u)$ and $\phi''(u)$, continuous and single valued in T, and if $\phi''(u)$ is negative at all points of T, and if also, on C, $U - \phi\ (u) = 0$, where U is harmonic in T, then $U - \phi(u)$ is negative at all interior points of T. Let us prove the more general theorem:

If $\phi(u, v)$ is with its first and second partial derivatives continuous and single valued in T, and if $\Delta\phi$ is negative at all points of T, and if also, on C, $U - \phi(u, v) = 0$, where U is harmonic in T, then $U - \phi(u, v)$ is negative at all interior points of T.

Let $V = U - \phi(u, v)$. Then, at all points of T,

$$\Delta V = -\Delta\phi > 0.$$

Since V is zero on C, if it is positive at any interior point of T, there is within T a maximum value of V where necessarily

$$\frac{\partial^2 V}{\partial u^2} < 0, \qquad \frac{\partial^2 V}{\partial v^2} < 0.$$

These conditions are inconsistent with the fact that ΔV is positive at all points of T. If at any interior point V vanishes, since no displacement to a neighboring interior point can cause V to increase, we must have at this point

$$\frac{\partial V}{\partial u} = \frac{\partial V}{\partial v} = 0,$$

and then for the same reason

$$\frac{\partial^2 V}{\partial u^2} \leqq 0 \qquad \frac{\partial^2 V}{\partial v^2} \leqq 0,$$

conditions again impossible.

To apply these theorems to the proof of Tchébychef's principle, let

$$V = \log m = U - \tfrac{1}{2} \log E.$$

Then is

$$\phi(u, v) = \tfrac{1}{2} \log E, \qquad \Delta\phi = -EK.$$

Hence, if the region of the surface to be mapped is without singularities and of positive curvature throughout, the conditions of this theorem are satisfied. This is indeed the case with any region of the spheroid.

Suppose now that U has been found so that, on C, V is constant. To suppose this constant zero will affect only the size of the map, and be no essential restriction. Then suppose, on C,

$$V = \log m = U - \tfrac{1}{2} \log E = 0.$$

By the theorem just proved V is negative at all interior points of T. Let D represent the "maximum maximorum" of the absolute value of V at all points of T. Then D is the *deviation* of the map in the $u'v'$ plane, S'. The deviation in Tchébychef's principle is the difference between the greatest and least values of the logarithm of the scale, or the logarithm of the ratio of the greatest value of the scale to its least value. We may now prove that the deviation of V is less than that of V_1, where

$$V_1 = U_1 - \tfrac{1}{2} \log E,$$

and U_1 is any harmonic function in T not differing by a constant from U.

Let the maximum value of V_1 on C be a. Then $V_1 - a$ has the same deviation as V_1, and this difference is at some point of C zero, at some points certainly negative, since $U_1 - U$ is not constant, but at no point of C positive. We may write

$$V_1 - a = V + U_1 - a - U,$$

and since V vanishes at all points of C, the harmonic function $U_1 - a - U$ is zero and negative but not positive on C. Then $U_1 - a - U$ must be negative at all interior points of T, and since the same is true of V, the sum, $V_1 - a$, which vanishes at some point of C, has within T a negative value numerically greater than V. Then $V_1 - a$, and consequently V_1, have a greater deviation than V.

We may note that a map of a region of positive curvature constructed on this principle has its greatest scale on the boundary. A similar discussion shows that a map of a region of negative total curvature at each point, constructed on this principle, has a minimum deviation, and has its least scale on the boundary.

We now take up the proof of Eisenlohr's theorem. We make our point of departure two theorems concerning geodesic curvature:

1. The geodesic curvature of a geodesic line of a surface is at every point zero.

2. If the surface is plane the geodesic curvature of any curve on the surface is the ordinary curvature. It may be proved that, when the linear element of a surface is given by $ds^2 = E(du^2 + dv^2)$, the geodesic curvature, $1/\rho_\phi$, of the curve on the surface whose equation is $\phi(u, v) = 0$ is given by*

$$-\frac{1}{\rho_\phi} = \frac{1}{E}\left\{\frac{\partial}{\partial u}\frac{\sqrt{E}\,\frac{\partial\phi}{\partial u}}{\sqrt{\left(\frac{\partial\phi}{\partial u}\right)^2+\left(\frac{\partial\phi}{\partial v}\right)^2}} + \frac{\partial}{\partial v}\frac{\sqrt{E}\,\frac{\partial\phi}{\partial v}}{\sqrt{\left(\frac{\partial\phi}{\partial u}\right)^2+\left(\frac{\partial\phi}{\partial v}\right)^2}}\right\}.$$

Suppose a map constructed in the plane S', whose linear element is given by $ds'^2 = E'(du^2 + dv^2)$, where as in the previous discussion

$$u' \pm iv' = f(u + iv) \quad \text{and} \quad E' = f'(u + iv)\,\bar{f}'(u - iv).$$

Let us introduce a variable, τ, defined by the equations

$$\frac{\frac{\partial\phi}{\partial u}}{\sqrt{\left(\frac{\partial\phi}{\partial u}\right)^2+\left(\frac{\partial\phi}{\partial v}\right)^2}} = \cos\tau, \qquad \frac{\frac{\partial\phi}{\partial v}}{\sqrt{\left(\frac{\partial\phi}{\partial u}\right)^2+\left(\frac{\partial\phi}{\partial v}\right)^2}} = \sin\tau.$$

Then we may write for a geodesic line of the surface, whose equation is $\phi(u, v) = 0$, and for the curvature, $1/\rho$, of its representation on the map S', which has the same equation,

$$0 = \frac{1}{E}\left\{\frac{\partial}{\partial u}(\sqrt{E}\,\cos\tau) + \frac{\partial}{\partial v}(\sqrt{E}\,\sin\tau)\right\},$$

$$-\frac{1}{\rho} = \frac{1}{E'}\left\{\frac{\partial}{\partial u}(\sqrt{E'}\cos\tau) + \frac{\partial}{\partial v}(\sqrt{E'}\sin\tau)\right\}.$$

If we perform the indicated differentiations, we may eliminate the derivatives of τ, and obtain the result

$$-\frac{1}{\rho} = \frac{1}{\sqrt{E'}}\left\{\left(\frac{1}{\sqrt{E'}}\frac{\partial\sqrt{E'}}{\partial u} - \frac{1}{\sqrt{E}}\frac{\partial\sqrt{E}}{\partial u}\right)\cos\tau + \left(\frac{1}{\sqrt{E'}}\frac{\partial\sqrt{E'}}{\partial v} - \frac{1}{\sqrt{E}}\frac{\partial\sqrt{E}}{\partial v}\right)\sin\tau\right\}.$$

* For proof of this formula and the statements concerning geodesic curvature see Bianchi-Lukat, *Differentialgeometrie*, chap. VI.

The maximum value at a point (u, v) of $1/\rho^2$, found by varying τ, is easily seen to be

$$\frac{1}{E'}\left(\frac{1}{\sqrt{E'}}\frac{\partial\sqrt{E'}}{\partial u}-\frac{1}{\sqrt{E}}\frac{\partial\sqrt{E}}{\partial u}\right)^2+\frac{1}{E'}\left(\frac{1}{\sqrt{E'}}\frac{\partial\sqrt{E'}}{\partial v}-\frac{1}{\sqrt{E}}\frac{\partial\sqrt{E}}{\partial v}\right)^2,$$

which we will write $1/r^2$, and which may be transformed into the expression

$$\frac{1}{r^2}=\frac{E}{E'^2}\left\{\left(\frac{\partial}{\partial u}\sqrt{\frac{E'}{E}}\right)^2+\left(\frac{\partial}{\partial v}\sqrt{\frac{E'}{E}}\right)^2\right\}.$$

We have now the scale of the map given by

$$m=\frac{ds'}{ds}=\sqrt{\frac{E'}{E}},\text{ and consequently }\frac{E}{E'^2}=\frac{1}{E'm^2},$$

so that we may write

$$\frac{1}{r^2}=\frac{1}{E'm^2}\left\{\left(\frac{\partial m}{\partial u}\right)^2+\left(\frac{\partial m}{\partial v}\right)^2\right\}=\frac{1}{E'}\left\{\left(\frac{\partial\log m}{\partial u}\right)^2+\left(\frac{\partial\log m}{\partial v}\right)^2\right\}.$$

The harmonic function U is to be chosen so that the mean value of $1/r^2$ on the map shall be a minimum. We must then make a minimum the integral over the map

$$\iint\frac{du'\,dv'}{r^2}=\iint\frac{E'\,du\,dv}{r^2}=\iint\left\{\left(\frac{\partial\log m}{\partial u}\right)^2+\left(\frac{\partial\log m}{\partial v}\right)^2\right\}du\,dv.$$

Now we have $\log m = U - \frac{1}{2}\log E$, and E is a given function of u and v so that any variation of $\log m$ is a variation of U alone and consequently itself a harmonic function. Let any such variation be represented by q. Then a necessary condition that the integral be a minimum is the vanishing of the first variation of the integral so that

$$\iint\left[\frac{\partial\log m}{\partial u}\frac{\partial q}{\partial u}+\frac{\partial\log m}{\partial v}\frac{\partial q}{\partial v}\right]du\,dv=0.$$

This integral may, by Green's theorem,* be written

$$\int \log m \frac{\partial q}{\partial n} \cdot ds - \int\int \log m \cdot \Delta q \cdot du \cdot dv = 0,$$

where n denotes the exterior normal and the simple integral is taken around the boundary of the map in the uv plane, S_1. Now since q is a harmonic function, Δq vanishes, and the condition becomes

$$\int \log m \cdot \frac{\partial q}{\partial n} \cdot ds = 0.$$

We know that for any harmonic function the integral

$$\int \frac{\partial q}{\partial n} ds = 0,$$

and from this and the condition found Eisenlohr infers immediately that $\log m$ must be constant on the boundary. This conclusion does not appear to me wholly obvious, but may be reached, when the map is simply connected, as follows: Let the harmonic function conjugate to q be p. Then we know

$$\frac{\partial q}{\partial n} = \frac{\partial p}{\partial s}$$

and

$$\int \log m \frac{\partial q}{\partial n} \cdot ds = \int \log m \frac{\partial p}{\partial s} \cdot ds$$

$$= \int \left\{ \frac{\partial (p \log m)}{\partial s} - p \frac{\partial \log m}{\partial s} \right\} ds.$$

If the map on the plane S_1 is simply connected, p is single valued† and

$$\int \frac{\partial (p \log m)}{\partial s} ds = 0.$$

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* See, for example, Picard, *Traité d'analyse*, vol. 2, p. 10.

† Osgood, *Functionentheorie*, p. 546.

In that case the condition becomes

$$\int p \frac{\partial \log m}{\partial s} ds = 0.$$

Again if the region is simply connected a harmonic function p may be found taking any continuous set of boundary values,* and the variation q chosen as the single valued function conjugate to p, and consequently the condition requires that at every point of the boundary

$$\frac{\partial \log m}{\partial s} = 0,$$

and m must be constant on the boundary. It is to be noticed that this condition is not proved sufficient to make the integral a minimum, and that consequently the principle of Eisenlohr is less complete than that of Tchébychef.

* By the solution of "Dirichlet's problem." See Picard, loc. cit., vol. 2, p. 38. We must require further that the boundary consist of a finite number of "regular" arcs.

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APPLICATION OF TCHÉBYCHEF'S PRINCIPLE IN THE PROJECTION OF MAPS*

By G. W. Hill

The question how to obtain the best map of a portion of the surface of a sphere or ellipsoid of revolution is certainly an important one. Tchébychef was, I believe, the first mathematician to consider this matter from a general point of view. His dictum is "A map is the best when, within its limits, the logarithm of the scale has the least possible oscillation." It is here assumed that this function of the scale is a measure of its own importance. It may be objected that the function enjoying this property is not the logarithm. This, however, has no more validity than the objections made against the method of least squares in the treatment of discordant observations. The author evidently was induced to adopt his principle by the elegant transformation it is capable of.

In the projection of the oblate ellipsoid of revolution on a plane, let x and y denote the rectangular coordinates of the projected point, u the longitude and z the geographical colatitude of the point to be projected. Let ϵ denote the eccentricity of the meridian of the ellipsoid. To avoid the consideration of a linear unit we assume that the ellipsoid has the semi-axis 1, and that the scale on the outer contour of the map is also 1. The multiplications to be made for other conditions are so obvious that they need no mention. As we are not restricted to the use of z and u for defining the position of a point on the ellipsoid, and as the relative properties of this surface and the plane, in respect to projection, demand another variable in the place of z, let us put

$$v = \log\left[\left(\frac{1 + \epsilon \cos z}{1 - \epsilon \cos z}\right)^{\frac{\epsilon}{2}} \tan \frac{z}{2}\right],$$

where the logarithm is hyperbolic. While z goes from 0 to π, v goes from $-\infty$ to $+\infty$. Then, according to the theory of *conforme* projection, the connection between the variables x, y and u, v is shown by the equation

* See the preceding paper.

$$x + y\sqrt{-1} = f(u + v\sqrt{-1}),$$

f denoting any arbitrary function. The scale m is given by the equation

$$\log m = \log \frac{\sqrt{1 - \epsilon^2 \cos^2 z}}{\sin z} + \tfrac{1}{2} \log f'(u + v\sqrt{-1}) + \tfrac{1}{2} \log f'(u - v\sqrt{-1}),$$

where f' denotes the derivative function of f.* For brevity let $\log m$ be denoted by V and its first term by A, and the two following terms involving the arbitrary function by W, so that

$$V = A + W.$$

In establishing his principle Tchébychef compares it to the statical case of the convection of heat. But we may, with advantage, substitute for the latter the general principle of hydrostatics.

Let V_1 denote the greatest value assumed by V in the portion of the surface to be projected, and V_0 the least. Then, in order that the map may be the best possible, $V_1 - V_0$ must be a minimum. This is expressed in the language of the calculus of variations by $\delta(V_1 - V_0) = 0$, the variation having reference to the form of the function f. (The criterion for discriminating a minimum from a maximum need not be noticed).

Now, to form the cognate problem in hydrostatics, let us suppose that, in the plane of u, v, the surface to be projected is covered with a fluid of infinitesimal depth, such that the gravitational action of itself on itself has a potential V. The fluid can be in equilibrium if V is constant along the boundary of the surface. In case the boundary consists of several detached portions, the theorem still holds, but the constants belonging to each portion may be different. Appealing to the principle of least action for the condition of equilibrium, if V_1 is the maximum of the potential and V_0 the minimum, just as before, $\delta(V_1 - V_0) = 0$, the δ having reference to the distribution of the density ρ of the fluid. Now Poisson's equation is

$$\frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial v^2} + 4\pi\rho = 0.$$

* For demonstration of these equations (see the preceding paper) consult Gauss, *Werke*, vol. 4, p. 193. The essential parts of Gauss' investigation have been translated by Thomas Craig in his *Treatise on Projections* (separate publication of the U. S. Coast and Geodetical Survey, Washington, 1882). At the end of the supplementary volume of Boole's *Differential Equations* will be found a brief treatment of the matter.

But, in the first problem, we have

$$\frac{\partial^2 V}{\partial u^2} + \frac{\partial^2 V}{\partial v^2} = \frac{\partial^2 A}{\partial u^2} + \frac{\partial^2 A}{\partial v^2}.$$

Hence, if we make

$$4\pi\rho = -\left(\frac{\partial^2 A}{\partial u^2} + \frac{\partial^2 A}{\partial v^2}\right),$$

the two problems are rendered identical.

Thus the transformation of Tchébychef's dictum is "That map is best in which the scale is constant along its boundary; it being understood that when the latter consists of disconnected portions the constant may vary from one portion to another."

The Tchébychef principle demands that, with our conventions, V should vanish on the contour of the map. It has become the fashion recently to allow meridians and parallels to form this contour. Consequently it is proposed to treat here the simple case where the portion of the ellipsoidal surface to be projected is contained between portions of two parallels and two meridians. By counting the longitudes from the middle meridian, we can suppose that $\mp u_0$ are the limiting values of u; also let v_0 and v_1 be the lower and upper limits of v; and z_0, z_1 the corresponding values of z. The problem, thus limited, has the advantage that the contour is regarded as dependent on u, v instead of x, y.

A is stated in terms of z instead of v; the first should be replaced by the second; but the equation connecting them is eminently transcendental, the infinite series in powers of ϵ^2 is complicated. Limiting ourselves to the first power,

$$\log \frac{\sqrt{1 - \epsilon^2 \cos^2 z}}{\sin z} = \log \cosh v + \epsilon^2 \tanh v (1 - \tfrac{1}{2} \tanh v).$$

It is readily seen that our problem is a case of what is known as the Dirichlet problem. In the mode of solving this, started by Fourier, it is necessary to suppose that

$$\log \frac{\sqrt{1 - \epsilon^2 \cos^2 z}}{\sin z} = A + Bv + \sum_{i=1}^{i=\infty} N_i \sin i \frac{\pi(v - v_0)}{v_1 - v_0},$$

A, B, N_i being constants. A and B are evidently determined by the

equations

$$A + Bv_0 = \log \frac{\sqrt{1 - \epsilon^2 \cos^2 z_0}}{\sin z_0}, \qquad A + Bv_1 = \log \frac{\sqrt{1 - \epsilon^2 \cos^2 z_1}}{\sin z_1}.$$

The expression

$$W = \tfrac{1}{2} \log f'(u + v\sqrt{-1}) + \tfrac{1}{2} \log f'(u - v\sqrt{-1})$$

satisfies the differential equation

$$\frac{\partial^2 W}{\partial u^2} + \frac{\partial^2 W}{\partial v^2} = 0.$$

A particular integral of this is $W = UV$, where U is a function of u alone, and V of v alone. By the substitution of this the differential equation becomes

$$V \frac{\partial^2 U}{\partial u^2} + U \frac{\partial^2 V}{\partial v^2} = 0.$$

This may be satisfied in two different ways, shown by the equations

$$\frac{\partial^2 U}{\partial u^2} = \pm c U, \qquad \frac{\partial^2 V}{\partial v^2} = \mp c V,$$

where c is a constant. Thus both U and V are periodic functions, one to a real, the other to an imaginary period. Which is to have the real, and which the imaginary period, depends on the circumstances of the problem. Here U is to have the imaginary and V the real period. Thus the particular integral may be written

$$W = M \cosh(cu + k) \sin(cv + h),$$

where M, k, h are the arbitrary constants. As the equation is linear, a more general integral may be written as

$$W = \sum [M \cosh(cu + k) \sin(cv + h)],$$

where the summation extends to all permissible values for c. But, in this case, W must have the same value when the sign of u is reversed, thus $k = 0$; also W must vanish when $v = v_0$ or v_1; this demands that c must be an integral multiple of $\pi/(v_1 - v_0)$. Also it is lawful to add $-A - Bv$ to the expression for W. All conditions will be fulfilled if we write

$$W = -A - Bv + \sum_{i=1}^{i=\infty} M_i \cosh(icu)\sin[ic(v - v_0)].$$

Here we have $c = \pi/(v_1 - v_0)$, and A and B have the constant values previously attributed to them.

Thus the proper form for $\log m$ is

$$\log m = \sum_{i=1}^{i=\infty} \left\{ N_i \sin ic(v - v_0) + M_i \cosh(icu) \sin ic(v - v_0) \right\}.$$

Thus $\log m$ clearly reduces to 0 when $v = v_0$ or v_1; and it will reduce to 0 when $u = u_0$, provided that M_i is determined by the equation

$$M_i \cosh(icu_0) = -N_i.$$

Hence

$$\log m = \sum_{i=1}^{i=\infty} \left[1 - \frac{\cosh(icu)}{\cosh(icu_0)}\right] N_i \sin[ic(v - v_0)].$$

By equating the two expressions for W we have

$$\tfrac{1}{2}\log f'(u + v\sqrt{-1}) + \tfrac{1}{2}\log f'(u - v\sqrt{-1}) = -A - Bv + \sum_{i=1}^{i=\infty} M_i \cosh(icu)\sin[ic(v - v_0)].$$

In attempting to derive the form of the function f from this equation some difficulties have been met with, and a method, suggested by a graphical treatment, has been substituted. The exposition of this is better understood in the elaboration of an example.

For this let us assume $z_0 = 40°$, $z_1 = 50°$, $u_0 = 5°$. The elements of the Clark spheroid give $\log_{10}\epsilon = 8.916\ 3655\ 961$. As showing the connection between the variables z and v we employ the equation

$$v = \log\tan\frac{z}{2} + \epsilon^2\cos z + \tfrac{1}{3}\epsilon^4\cos^3 z + \tfrac{1}{5}\epsilon^6\cos^5 z + \cdots.$$

Thence it results that

$$v_0 = -1.00547\ 07238, \quad v_1 = -0.75853\ 60483, \quad \log_{10} c = 1.10456\ 77942,$$
$$c = 12.72236\ 3332, \quad c \times 1^0 = 0.22204\ 71288.$$

If s be an argument which goes from 0 to π when v goes from v_0 to v_1, we have the following table of correspondence between v and z:

$12s$	v	z	$\log\frac{\sqrt{1-\varepsilon^2\cos^2 z}}{\sin z}$	$A + Bv$	Difference
0	−1.00547 07238	40°	0.43994 0702	0.43994 0702	0
π	98489 28342	40 45′57″.7853	42426 5941	42537 1276	−0.00110 5335
2π	96431 49446	41 32 39 .2681	40877 2264	41080 1851	202 9587
3π	94373 70549	42 20 4 .6247	39346 5399	39623 2425	276 7026
4π	92315 91653	43 8 14 .0019	37835 1192	38166 3000	331 1808
5π	90258 12757	43 57 7 .5135	36343 5594	36709 3574	365 7980
6π	88200 33861	44 46 45 .2414	34872 4649	35252 4149	379 9500
7π	86142 54964	45 37 7 .2306	33422 4519	33795 4723	373 0204
8π	84084 76068	46 28 13 .4891	31994 1444	32338 5297	344 3853
9π	82026 97172	47 20 3 .9873	30588 1761	30881 5872	293 4111
10π	79969 18276	48 12 38 .6544	29205 1884	29424 6446	219 4562
11π	77911 39379	49 5 57 .3775	27845 8308	27967 7021	−0.00121 8712
12π	75853 60483	50	26510 7595	26510 7595	0

The values of A and B are, $A = -0.2719\ 4625$, $B = -0.7080\ 1360$. The numbers in the last column of the table are to be represented by a periodic function of the argument s, and we find

$$\begin{aligned}
&-0.00392\ 1901 \sin s + 0.00008\ 6185 \sin 2s\\
&-0.00014\ 5846 \sin 3s + 0.00001\ 0693 \sin 4s\\
&-0.00003\ 1124 \sin 5s + 0.00000\ 3086 \sin 6s\\
&-0.00001\ 0791 \sin 7s + 0.00000\ 1187 \sin 8s\\
&-0.00000\ 4296 \sin 9s + 0.00000\ 0443 \sin 10s\\
&-0.00000\ 1185 \sin 11s.
\end{aligned}$$

The values of the common logarithms of the hyperbolic cosine of various multiples of $c \times 1^0$ are given in the following table:

i		i		i		i	
1	0.0106195	9	0.5747815	20	1.6277072	32	2.7848529
2	0.0414879	10	0.6683966	21	1.7241194	33	2.8812866
3	0.0899754	11	0.7630127	22	1.8205394	35	3.0741546
4	0.1526127	12	0.8582766	24	2.0133924	36	3.1705884
5	0.2258977	14	1.0499092	25	2.1098226	40	3.5563233
6	0.3068064	15	1.1460329	27	2.3026865	44	3.9420586
7	0.3929821	16	1.2422634	28	2.3991193	45	4.0384924
8	0.4827065	18	1.4349258	30	2.5919860	50	4.5206616
						55	5.0028307

The values of the latter factors of the terms of $\log m$ are easily derived from

$c(v - v_0)$	$c(v - v_0)$	$c(v - v_0)$	$c(v - v_0)$
0°	57° 27′ 5″.44	111° 44′ 32″.24	163° 20′ 3″.70
19 32′ 3″.44	75 52 30.43	129 13 8 .65	180
38 40 39 .54	93 57 58.82	146 24 42 .19	

The scale along the contour of the map being 1.000 0000, the deviations of it for each degree of colatitude and longitude within the area of the map in units of the seventh decimal are given in the following table:

z	$u = 0°$	$u = 1°$	$u = 2°$	$u = 3°$	$u = 4°$	$u = 5°$
40°	0	0	0	0	0	0
41	− 9820	− 9676	− 9212	− 8307	− 6618	0
42	16882	16618	15775	14155	11175	0
43	21685	21334	20226	18112	14254	0
44	24485	24088	22824	20418	16054	0
45	25377	24956	23646	21152	16627	0
46	24433	24041	22789	20402	16084	0
47	21625	21285	20200	18114	14291	0
48	16758	16505	15692	14119	11191	0
49	− 9669	− 9534	− 9097	− 8240	− 6604	0
50	0	0	0	0	0	0

The corresponding table for all values of u, in the case of Species II of Lagrange's projection, when the exponent is adopted so that the scale is equal along the northern and southern limits of the map, stands thus:

z		z		z		z	
40°	0	43°	− 32283	46°	− 36233	49°	− 13395
41	− 14032	44	36659	47	31538	50	0
42	− 24766	45	− 37956	48	− 23914		

The sum of all the deviations in the case of the map on Tchébychef's principle is 0.0754149: but, in the latter table the corresponding sum is 0.1504656: thus nearly double the previous one. Hence it may be judged how much the first map is to be preferred.

The method of constructing the map on Tchébychef's principle, here employed, depends on drawing each meridian and parallel at intervals of 1°. In the first place it is necessary, from the preceding table to derive the mean value of the scale in passing over a degree of longitude or colatitude. The rule adopted for finding this is to interpolate into the middle, and, differencing the resulting table, add one-sixth the second difference. Between $u = 4°$ and $u = 5°$ the change of differences is very rapid, and one must find means of estimating the mean independent of this rule. We have:

Values of the Mean Scale along Degrees of the Meridians

z	$u = 0°$	$u = 1°$	$u = 2°$	$u = 3°$	$u = 4°$
40°	0.9994875	94949	95187	95649	96530
41	86426	86632	87293	88574	90942
42	80537	80847	81832	83717	87168
43	76741	77118	78313	80590	84731
44	74928	75335	76638	79103	83577
45	74944	75349	76643	79100	83549
46	76810	77179	78356	80609	84709
47	80634	80934	81891	83738	87146
48	86576	86772	87404	88633	90946
49	94994	95063	95284	95722	96563
50					

VALUES OF THE MEAN SCALE ALONG DEGREES OF THE PARALLELS

z	$u = 0°$	$u = 1°$	$u = 2°$	$u = 3°$	$u = 4°$
41°	0.9990227	90524	91189	92504	95485
42	83206	83748	84946	87195	92354
43	78432	79147	80716	83598	90309
44	75647	76460	78263	81527	89120
45	74763	75622	77474	80885	88830
46	75698	76502	78290	81528	89277
47	78488	79184	80729	83603	90282
48	83326	83846	85019	87202	92407
49	90376	90754	91288	92535	95597

According to the dimensions of the Clark spheroid the curved distances along the meridians and parallels in miles are:

z	Meridians	Parallels	Reduced to Chords on the Map						
			0°	1°	2°	3°	4°	5°	
40°		m 44.55229	m 69.10934	34	33	33	32	31	m 44.55201
41	m 69.10934	45.46947							45.46919
42	69.09711	46.37268	69.09711	11	10	10	09	08	46.37239
43	69.08483	47.26165	69.08483	83	82	82	81	80	47.26136
44	69.07251	48.13610	69.07251	51	50	50	49	48	48.13580
45	69.06017	48.99579	69.06017	17	16	16	15	14	48.99549
46	69.04784	49.84045	69.04784	84	83	83	82	81	49.84014
47	69.03551	50.66984	69.03551	51	50	50	49	48	50.66953
48	69.02320	51.48370	69.02320	20	19	19	18	17	51.48338
49	69.01093	52.28180	69.01093	93	92	92	91	90	52.28148
50	68.99873	53.06389	68.99873	73	72	72	71	70	53.06356

The reductions from the arcs to the chords are very minute, at most only 33 units of the last decimal for the parallels and 3 units for the meridians.

Desiring to construct the map on a scale of 100^m to the inch, we have only to divide the numbers just given by 100 and multiply by the mean scale to get the distances between the projected points on the map. Thus is obtained the following table:

DISTANCES ALONG THE MERIDIANS

z	$u = 0°$	$u = 1°$	$u = 2°$	$u = 3°$	$u = 4°$	$u = 5°$
40°	^{m}0.6907394	7444	7606	7927	8533	.0931
41	6900332	0475	0930	1814	3451	9708
42	6895036	5251	5932	7233	9616	.8480
43	6891184	1444	2270	3843	6703	.7248
44	6888703	8984	9882	.1584	.4673	.6014
45	6887484	7763	8654	.0352	.3424	.4781
46	6887541	7797	8606	.0163	.2992	.3548
47	6888954	9160	9819	.1094	.3444	.2317
48	6891829	1965	2398	3248	4841	.1090
49	6896419	6467	6617	6919	7500	9870
50						

DISTANCES ALONG THE PARALLELS

z	$u = 0°$	$u = 1°$	$u = 2°$	$u = 3°$	$u = 4°$
40°	^{m}0.4455201	55201	55201	55201	55201
41	4542504	42639	42941	43540	44895
42	4629481	29731	30287	31330	33722
43	4715942	16280	17022	18383	21555
44	4801857	02249	03118	04648	08343
45	4887184	87605	88512	90184	94075
46	4971901	72302	73194	74807	78659
47	5056053	56406	57190	58645	62029
48	5139752	40021	40625	41748	44428
49	5223117	23314	23588	24246	28547
50	5306356	06356	06356	06356	06356

We have now the sides of 50 quadrilaterals, which must be so disposed that they continuously cover the plane of projection, and moreover, must fit against one of the projected parallels. We take for the latter the one which corresponds to $z = 50°$. By subtracting the distances along the parallel of $z = 49°$ from those of $z = 50°$ we obtain the quantities in the first column of the following table, and, by comparing them with the distances along the meridians, we obtain the inclinations of two adjacent meridians at the beginning of their courses. By subjecting these to integration we have the inclination of each meridian to the middle meridian. Multiplying the sine and cosine of this by 0.5306356 we have the increments of the coordinates x and y which we denote by Δx and Δy. Thus we have the following table:

u	Differences	Inclination	Integrated	Δx	Δy
1°	0.0083239	41′ 29″.46	0° 20′ 44″.73	+ 0.0032022	+ 0.5306260
2	83042	41 23 .52	1 2 11 .22	95984	5305488
3	82768	41 15 .21	1 43 30 .58	159749	5303951
4	82110	40 55 .32	2 24 35 .85	223127	5301663
5	80509	40 6 .62	3 5 6 .82	285595	5298665

Having now the coordinates of angles of the quadrilaterals which abutt against the parallel of $z = 50°$, all the quadrilaterals can be made to fill continuously the plane of projection in determinate fashion. The rigorous following out of this procedure requires the determination of one of the diagonals of each quadrilateral, whence the position of the two unknown sides is derived. But the little curvature in the meridians enables us to greatly shorten the labor. Each meridian may be regarded as the envelop of all the circles described from the known corners of the quadrilaterals just beneath it with the appropriate radii given in a preceding table. Thus it is easy to compute the inclination of the two sides of each quadrilateral which form parts of two consecutive meridians, and by addition to get the inclination on the middle meridian. Thence are determined the changes of the coordinates in passing along the meridian from one parallel to the following. The table shows these inclinations and the values of Δx and Δy:

	$u = 1°$			$u = 2°$		
z	Inclination	Δx	Δy	Inclination	Δx	Δy
50°						
49	41′ 29″.46	+.′6895963	−.0083233	1° 22′ 52″.98	+.6894613	−.0166259
48	41 34 .86	6891460	83359	23 7 .41	6890387	166640
47	41 45 .87	6888652	83693	23 28 .99	6887787	167298
46	41 59 .92	6887283	84146	23 58 .12	6886561	168241
45	42 16 .86	6887243	84710	24 35 .78	6886568	169499
44	42 34 .69	6888457	85321	25 9 .39	6887768	170699
43	42 51 .35	6890911	85908	25 44 .01	6890127	171868
42	43 6 .26	6894710	86454	26 15 .38	6893762	173007
41	43 19 .72	6899927	86970	26 42 .71	6898733	174047
40	43 26 .83	6906892	87296	26 57 .63	6905397	174715

	$u = 3°$			$u = 4°$		
z	Inclination	Δx	Δy	Inclination	Δx	Δy
50°						
49	2° 4′ 8″.19	+.6892424	−.0248993	2° 45′ 3″.51	+.6889551	−.0331046
48	4 29 .77	6888727	249581	45 37 .65	6886840	332059
47	5 6 .25	6886531	250721	46 32 .74	6885356	333830
46	5 52 .52	6885544	252231	47 41 .16	6884795	336092
45	6 50 .64	6885662	254179	49 2 .60	6885090	338832
44	7 48 .60	6886822	256159	50 27 .42	6886200	341725
43	8 39 .89	6889014	257956	51 39 .75	6888106	344241
42	9 29 .09	6892341	259727	52 51 .42	6890897	346781
41	10 12 .96	6896865	261366	53 55 .85	6894616	349127
40	10 37 .33	6902940	262413	54 35 .68	6899624	350717

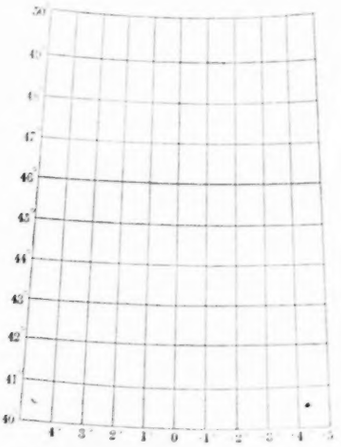
	$u = 5°$		
z	Inclination	Δx	Δy
50°			
49	3° 25′ 10″.13	+ .6887585	−.0411548
48	26 11 .05	6888681	413655
47	27 34 .99	6889739	416533
46	29 11 .98	6890770	419847
45	31 9 .23	6891760	423840
44	33 7 .89	6892746	427881
43	34 51 .29	6893762	431423
42	36 33 .69	6894776	434913
41	38 7 .31	6895803	438120
40	39 12 .55	6896886	440379

By the addition of the values of Δx and Δy we arrive at the following values of the rectangular coordinates of the intersections of the meridians and parallels:

	$u = 0°$		$u = 1°$		$u = 2°$	
Lat.	x	y	x	y	x	y
	in		in	in	in	in
40°	0.000000	0	0.003202	0.530626	0.012801	1.061175
41	0.689642	0	0.692798	522303	0.702262	1.044549
42	1.378825	0	1.381944	513967	1.391301	1.027885
43	2.067720	0	2.070810	505597	2.080079	1.011155
44	2.756474	0	2.759538	497183	2.768735	0.994331
45	3.445223	0	3.448262	488712	3.457392	0.977381
46	4.134093	0	4.137108	480180	4.146169	0.960314
47	4.823211	0	4.826199	471589	4.835182	0.943127
48	5.512715	0	5.515670	462944	5.524558	0.925827
49	6.202748	0	6.205663	454247	6.214431	0.908422
50	6.893488	0	6.896352	445517	6.904971	0.890950

	$u = 3°$		$u = 4°$		$u = 5°$	
Lat.	x	y	x	y	x	y
40°	0.028775	1.591570	0.051088	2.121736	0.079648	2.651603
41	0.718018	1.566671	0.740043	2.088632	0.768406	2.610448
42	1.406891	1.541712	1.428727	2.055426	1.457274	2.569082
43	2.095544	1.516640	2.117263	2.022043	2.146248	2.527436
44	2.784098	1.491417	2.805742	1.988433	2.835325	2.485444
45	3.472664	1.465999	3.494251	1.954550	3.524501	2.443060
46	4.161346	1.440383	4.182871	1.920385	4.213776	2.400282
47	4.850248	1.414588	4.871682	1.885954	4.903152	2.357130
48	5.539482	1.388615	5.560772	1.851276	5.592630	2.313629
49	6.229168	1.362479	6.250233	1.816363	6.282210	2.269827
50	6.919462	1.336237	6.940196	1.781291	6.971898	2.225789

These numbers serve for plotting the appended map, which covering an area of about 700 miles by 500 miles, has a variation of scale of about a quarter of one per cent. The scale on the boundary is 300^m to the inch. Such a map is interesting as being the best possible if we admit Tchébychef's principle.



RESEAU OF MERIDIANS AND PARALLELS CONSTRUCTED ON TCHÉBYCHEF'S PRINCIPLE.

THE FOUNDATIONS OF TRIGONOMETRY

By Arthur C. Lunn

Introduction. In the existing literature of real analysis the purely logical introduction of the trigonometric functions, when done on the basis of theorems of analysis alone without appeal to any geometric setting to supply features of the proofs, seems to be accomplished essentially in two ways, each of which is interpretable in terms of either real or complex variables.

According to the first method the functions sine and cosine, in terms of which the others are defined, are introduced by definition in terms of the series

$$(1)\quad S(x) = x - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \frac{x^7}{\lfloor 7} + \cdots, \qquad C(x) = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \cdots.$$

Then the proof of the permanent convergence of these series shows the existence of the functions for every value of the argument, and the theory of infinite series furnishes the basis of the proof that they possess the properties ascribed by elementary trigonometry to the sine and cosine of an angle whose radian measure is x.

According to the second method these functions are distinguished as solution of the differential equations

$$(2)\qquad S'(x) = C(x), \qquad C'(x) = -S(x),$$

with the special conditions

$$S(0) = 0, \qquad C(0) = 1.$$

These four conditions may then be viewed as a set of postulates, easily shown to be independent, involving as undefined elements the functions S and C, whose existence and unique determination are shown by means of the theory of differential equations.

But these two methods are hardly as elementary as might be desired, implying as they do a developed theory, in the one case of infinite series, in the other of differential equations. Moreover, the connection with the intuitional origin of the functions out of geometric constructions is so remote, that it is not easy to decide what properties of these functions must be proved in the critical theory in order to identify them completely as the analytic equivalents of the sine and cosine as understood in the practical trigonometry of lines and angles. The object here is to offer a supplementary point of view, admitting logical development of equal rigor, but based more directly on

general theorems only of analysis, and more directly interpretable in terms of elementary trigonometric notions.

Part I gives a set of postulates chosen suitably from familiar trigonometric propositions, with proof that they determine uniquely the functions sine and cosine and the constant π. Part II shows how the known expressions for sine, cosine, and angle, in terms of the tangent of the half-angle as auxiliary, may be regarded as definitions which lend themselves readily to a proof of the existence of functions satisfying these postulates. Part III contains a proof of the independence of the postulates, admitting the logical consistency of the preceding theory. Attention is confined to the case of real variables.

I. Postulates and Proof of Uniqueness. Let $S(x)$ and $C(x)$ be symbols of two undefined single-valued functions of the real variable x, and p the symbol of an undefined constant. With the admission, as part of the logical background, of such propositions of real analysis as will be specified, directly or by implication, it will be shown that these functions and this constant are uniquely determined by the following postulates, which are thus sufficient, not only to serve as basis for the development of analytic trigonometry, but also in the sense that their justification by inference from diagrams would make the connection with geometric trigonometry complete. For brevity, P will be used to represent the set of all numbers of the form $ip/2^j$, where i, j are integers.

POSTULATES.

1. $S(x + y) = S(x)C(y) + C(x)S(y)$, *if* x, y *are numbers of the set* P.
2. $S(x)$ *is continuous for every* x.
3. $C(x)$ *is continuous for every* x.
4. $C(p/2) = 0$.
5. $S(p/2) = 1$.
6. $C(p) = -1$.
7. $C(x)$ *is not negative for any* x *of the set* P *on the interval* $0 \cdots p/2$.
8. *If* $S(x)/x$ *has a limit as* x *approaches zero over the set of numbers* $p/2^n$, n *being a positive integer, that limit is unity.*

The first postulate is the addition formula for the sine, stated, however, only for angles which can be constructed from the angle p (straight angle) by successive bisections and additions; the seventh may be said to identify $p/2$ as the angular measure of a quadrant, whatever be the value of p; while the

eighth, suggested by the construction of the sequence of inscribed polygons of 2^n sides, is needed to determine the number p and thus fix the unit of measure of the angles. The first seven postulates alone determine the functions $\sin(\pi x/p)$, $\cos(\pi x/p)$, leaving p arbitrary.

Since the value of a continuous function at any point is determined by its values for the points of any set everywhere dense, it will be sufficient to prove that the values of S and C are determined for every point of P, which is such a set. For the following proof of uniqueness then, unless otherwise stated, x, y will be understood to belong to P.

From postulates 1, 4, 5, we have

$$S(x + p/2) = C(x), \qquad S(p) = 0, \tag{3}$$

so that

$$C(x + p/2) = S(x + p),$$

which by the aid also of 6 becomes

$$C(x + p/2) = -S(x); \tag{4}$$

hence, by substitution of $x - p/2$ for x,

$$S(x - p/2) = -C(x), \quad C(x - p/2) = S(x). \tag{5}$$

These equations, with arguments on the two sides differing by $p/2$, give by induction all of the familiar formulae needed for "reduction to the first quadrant," and also by substitution of $x + y$ for x the addition formula for the cosine:

$$C(x + y) = S(x + y + p/2) = C(x)C(y) - S(x)S(y). \tag{6}$$

The values of S and C will therefore be determined everywhere by their values at the points of the set P_1, defined as the points of P which lie on the interval $0 \cdots p/2$, or "first quadrant." It may be noted that on account of the continuity assumed, all of the preceding formulae are immediately proved valid for any values of x and y.

Now let

$$\sigma(x) = \overline{S(x)}^2 + \overline{C(x)}^2;$$

then

$$\sigma(x + y) = \sigma(x)\,\sigma(y), \qquad \sigma(2x) = \overline{\sigma(x)}^2;$$

and since $\sigma(p) = 1$, the latter gives by induction $\sigma(p/2^n) = 1$, where n is any positive integer, there being no ambiguity of sign in the square root because σ is a sum of squares. Hence, by the addition formula for σ just

given, and by continuity, $\sigma(x) = 1$ for every x; that is,

$$\overline{S(x)}^2 + \overline{C(x)}^2 = 1.$$

On this account comes in a familiar way

$$C(2x) = \overline{C(x)}^2 - \overline{S(x)}^2 = 2\,\overline{C(x)}^2 - 1,$$

giving

$$\overline{C(x)}^2 = \frac{1 + C(2x)}{2}, \tag{7}$$

which, starting at $x = p/2$, determines C successively for all points of the set $p\,2^n$, the ambiguity of sign being removed by postulate 7. Then S is determined for the same set by

$$S(x) = \frac{S(2x)}{2\,C(x)}, \tag{8}$$

where, as appears from the preceding determination of $C(x)$, the denominator is not zero for any value of x concerned here. Then the addition formulae complete the determination of S and C for all points of the set P_1.

To show that the constant p is determined, let

$$x_i = \frac{p}{2^{i+1}}, \qquad S_i = S(x_i), \qquad C_i = C(x_i), \qquad L_i = \frac{S_i}{x_i}.$$

Then the recursion formulae $x_{i-1} = 2x_i$, $S_{i-1} = 2S_iC_i$ give $L_{i-1} = L_iC_i$, in which $L_0 = 2/p$; so that

$$\frac{p}{2}\,L_n = M_n = \prod_{i=1}^{n} \frac{1}{C_i}.$$

Here the numbers M_n, $n = 1, 2, 3, \cdots \infty$, form a monotonic increasing sequence, since every C_i is positive and less than unity. But the sequence is bounded: for

$$C_i^2 = \frac{1 + C_{i-1}}{2}, \qquad C_0 = 0,$$

which gives successively

$$\begin{aligned} C_1^2 &= \tfrac{1}{2}, & C_1 &> \tfrac{1}{2}; \\ C_2^2 &> \tfrac{3}{4}, & C_2 &> \tfrac{3}{4}; \\ &\cdot & &\cdot \\ &\cdot & &\cdot \\ &\cdot & &\cdot \\ C_i^2 &> 1 - \frac{1}{2^i}, & C_i &> 1 - \frac{1}{2^i}; \end{aligned}$$

and therefore

$$M_n < \frac{1}{\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{2^n}\right)} \leqq \frac{1}{\frac{1}{2}\left[1-\frac{1}{4}-\frac{1}{8}-\cdots-\frac{1}{2^n}\right]} \leqq 4.$$

By a fundamental proposition due to Weierstrass this sequence has therefore a definite finite limit; hence the sequence of values of L_n has a limit, which by postulate 8 is unity; and hence $p = \pi$, the definite number defined by

$$\frac{\pi}{2} = \lim_{n=\infty} M_n, \tag{9}$$

where

$$M_1 = \frac{1}{\sqrt{\frac{1}{2}}}, \qquad M_2 = \frac{1}{\sqrt{\frac{1}{2}}} \cdot \frac{1}{\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}},$$

$$M_3 = \frac{1}{\sqrt{\frac{1}{2}}} \cdot \frac{1}{\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}} \cdot \frac{1}{\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}}, \ldots .*$$

The proof of the uniqueness of the system determined by postulates 1–8 is thus complete.

The particular set of postulates here used is of course arbitrary, and subject to endless possible modifications without loss of its desired character. One change perhaps desirable would be to state the addition theorem or functional equation only for acute angles, and to supplement it by such conditions as to make the values of the functions in the other quadrants determinate. Another method in the same spirit as the above would be to state a set of postulates involving a single undefined function such as to determine it as the tangent of the angle, and then to define the sine and cosine in terms of it, or preferably in terms of the tangent of the half-angle so as to avoid irrational expressions.

II. Existence Proof. For the further development of the consequences of the above postulates the proof could run mainly along conventional lines, leading among other things to the equations (2) for the derivatives, and from these by Taylor's theorem to the series in (1). For the present purpose it may be noted that if t be the tangent of the half-angle, defined by

$$t = S(x/2)/C(x/2),$$

* Geometrically, M_n is one-fourth the perimeter of a regular polygon of 2^{n+1} sides, inscribed in a circle of unit radius.

there would emerge the formulae

$$S(x) = \frac{2t}{1+t^2}, \quad C(x) = \frac{1-t^2}{1+t^2}, \tag{10}$$

while the angle x would be related to t by a differential equation integrable by quadrature:

$$\frac{dt}{dx} = \frac{1+t^2}{2}.$$

These equations suggest a simple means of proving the existence of functions which satisfy the postulates of part I. For, let

$$I(v, w) = \int_v^w \frac{dz}{1+z^2}, \tag{11}$$

and let the number π be defined by

$$\pi = 2\, I(0, \infty), \tag{12}$$

which is finite; then by the Archimedean property of the linear continuum, any value x of the fundamental independent variable (angle) can be written in the form

$$x = \xi + 2r\pi, \tag{13}$$

where r is an integer, and ξ is a number, uniquely determined, satisfying the condition $-\pi < \xi \leqq \pi$. Moreover, since the function under the integral sign is always positive, the integral $I(0, t)$ is a monotonic continuous function of its upper limit t and ranges from $-\pi/2$ to $+\pi/2$ as t ranges from $-\infty$ to $+\infty$. The equation

$$\xi = 2I(0, t) \tag{14}$$

for any ξ numerically less than π thus determines t uniquely as a monotonic continuous function of ξ, and therefore of x for all values between any two successive odd multiples of π.

Equations (10) may then be used as indirect definitions of functions $S(x)$ and $C(x)$, making these exist as single-valued continuous functions of x (postulates 2 and 3) for all values of x except those of the form $(2r+1)\pi$. Continuity at these points and also satisfaction of postulate 6 are secured by interpreting the definition so as to imply also the limiting values $S = 0$, $C = -1$ for $t = \pm\infty$; for these limits are independent of the sign of t as its numerical value approaches infinity, this sign depending on whether x is above or below the "straight-angle" value concerned.

Now the substitution $z = 1/z'$ transforms the integral $I(0, 1)$ into $I(1, \infty)$, so that $\pi/2 = 2I(0, 1)$ since $\pi = 2\{I(0, 1) + I(1, \infty)\}$. This gives the related values: $x = \pi/2$, $t = 1$, and for these $S = 1$, $C = 0$, as in postulates 4 and 5. Moreover, as x ranges over the interval $0 \cdots \pi/2$, t ranges over the interval $0 \cdots 1$, hence C is not negative there (postulate 7).

If x is numerically less than π, the mean-value theorem applied to the integral $I(0, t)$ shows that x lies between $2t$ and $2t/(1 + t^2)$, so that $S(x)/x$ lies between $1/(1 + t^2)$ and 1; as x approaches 0, t must approach 0, therefore

$$\lim_{x=0} \frac{S(x)}{x} = 1,$$

a stronger form of postulate 8.

On account of continuity it is sufficient to verify the addition formula for angles x, y which are not integral multiples of π, so that they may be written

$$x = 2I(0, t) + 2r\pi, \qquad y = 2I(0, u) + 2s\pi, \tag{15}$$

where t, u, are finite. Now the substitution

$$z = \frac{z' - t}{1 + z't}, \qquad z' = \frac{z + t}{1 - zt}, \qquad \frac{dz}{1 + z^2} = \frac{dz'}{1 + z'^2}$$

transforms the integrand function of $I(0, u)$ into itself; but with respect to the limits of integration a distinction must be made depending on whether the point of discontinuity $z = 1/t$ occurs on the interval of integration; we have then

$$I(0, u) = \begin{cases} I(t, v) & \text{when } tu < 1, \\ I(t, \infty) & \text{when } tu = 1, \\ I(t, \infty) + I(-\infty, v) & \text{when } tu > 1, \end{cases}$$

where

$$v = \frac{t + u}{1 - tu}, \tag{16}$$

and corresponding to these respectively

$$x + y = \begin{cases} 2\,I(0, v) + 2(r + s)\,\pi, \\ (2r + 2s + 1)\,\pi, \\ 2\,I(0, v) + 2(r + s + 1)\,\pi. \end{cases} \tag{17}$$

The intermediate case where $tu = 1$ may be treated by means of limits, while the other two show that if $tu \neq 1$, then v as defined is that value of the parameter in the integral which corresponds to the sum of the angles. Hence

$$(18) \qquad S(x+y) = \frac{2 \cdot \dfrac{t+u}{1-tu}}{1+\left(\dfrac{t+u}{1-tu}\right)^2} = \frac{2t(1-u^2)+2u(1-t^2)}{(1+t^2)(1+u^2)},$$

or

$$S(x+y) = S(x)\,C(y) + C(x)\,S(y),$$

as required by postulate 1. This completes the verification of the existence of functions satisfying the postulates.

As an alternative method, the entire trigonometric theory could be based directly on the definitions contained in equations (10), (13), (14), and would be equivalent to any theory containing postulates 1–8 as propositions, since they were shown to be sufficient for unique determination of the undefined elements.

III. Proof of Independence. Let it now be granted that there exist in the realm of real analysis functions having the properties of the trigonometric sine and cosine, and also, for brevity, that a certain amount of the theory of the exponential function, which occurs in two of the examples to be given, has been deduced in the usual manner: then it remains to be shown that the postulates of part I are independent. According to a recognized method this is done by the following series of examples, each of which gives a possible interpretation for the symbols S, C, p, such as to violate the postulate corresponding to the number annexed while fulfilling all of the others. Details of verification need not for the most part be set down.

1. $S(x) = x\left(1 - \dfrac{2p-4}{p^2}\,x\right)$; $C(x) = 1 - \dfrac{2x}{p}$; p arbitrary. Here $S(x+y)$ and $S(x)C(y) + C(x)S(y)$ differ by $\dfrac{4xy}{p^2}\left\{2 - \dfrac{p-2}{p}(x+y)\right\}$, so that postulate 1 is violated; all the other postulates are satisfied.

2. $S(x) = \sin x$ for points of the set P, but having arbitrary values differing from those of $\sin x$ for other values of x; $C(x) = \cos x$ for every x; $p = \pi$. Note here that all the postulates aside from those of continuity refer only to the points of the set P.

3. Similar to 2, except that it is $C(x)$ which takes arbitrary values, differing from those of $\cos x$, for values of x outside of the set P.

4. $S(x) = a^x \dfrac{\sin 5x}{5}$; $C(x) = a^x \cos 5x$; $p =$ the smallest positive number satisfying the equation $\sin (5p/2) = 5/7$ (approximately $p = 0.318$); and $a = 7^{\frac{2}{p}}$. Here $C(p/2) = \sqrt{24}$.

5. $S(x) = \dfrac{\sin 2x}{2}$; $C(x) = \cos 2x$; $p = \dfrac{\pi}{2}$. Here $S(p/2) = 1/2$. The postulate concerned may be said to fix the vertical scale of the sine curve.

6. $S(x) = a^x \dfrac{\sin 2x}{2}$; $C(x) = a^x \cos 2x$; $p = \dfrac{\pi}{2}$; and $a = 2^{\frac{2}{p}}$. This makes $C(p) = -4$.

7. $S(x) = \sin x$; $C(x) = \cos x$; $p = 5\pi$. This brings positive and negative quarter-waves into the interval $0 \cdots p/2$ for the curve of $C(x)$.

8. $S(x) = \sin mx$; $C(x) = \cos mx$; $p = \dfrac{\pi}{m}$; where $m \neq 1$. Here $\lim\limits_{x=0} \dfrac{S(x)}{x} = m$. For example, m might be $\dfrac{\pi}{180}$, so that x would be measured in degrees.

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SUFFICIENT CONDITIONS FOR IMAGINARY ROOTS OF ALGEBRAIC EQUATIONS

By Otto Dunkel

Introduction. The study of algebraic equations has led to the development of a considerable number of conditions, necessary or sufficient or both, for the existence of imaginary roots of such equations. The theorems of Descartes, Budan-Fourier, Sturm, Newton-Sylvester,* which afford such conditions, are well known.

More recently Professor E. B. Van Vleck† has given a sufficient condition for the existence of the maximum number of imaginary roots; and Professor Kellogg‡ has given a necessary condition for the reality of all the roots, which is allied to a special case of a more general theorem which it is the purpose of this paper to develop. It will be shown that several well known conditions can be easily derived from this general theorem, which we shall refer to as the Fundamental Theorem; and in some instances sharper tests than those usually given can be obtained. One of the special cases of this theorem will be shown to be part of Newton's theorem on algebraic equations. The theorem itself permits not only of a generalization of some tests already known but also of the derivation of new tests.

Fundamental Theorem. The general equation of the n^{th} degree:

$$(1) \qquad f(x) \equiv p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0$$

will be considered; but for simplicity in the statement and also in the proof of the theorem it will be convenient to write the equation in the form:

$$(1') \quad a_0x^n + na_1x^{n-1} + \cdots + \frac{n!}{(n-i)!\,i!}a_ix^{n-i} + \cdots + na_{n-1}x + a_n = 0,$$

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* Cf. E. Netto, *Vorlesungen über Algebra*, vol. 1, for the statement of these theorems: Descartes' Theorem, p. 219; Budan-Fourier's Theorem, p. 216; Sturm's Theorem, p. 238; Sylvester's Theorem, p. 225; Newton's Theorem, p. 233.

† Van Vleck, Annals of Mathematics, ser. 2, vol. 4 (1903), p. 191.

‡ O. D. Kellogg, Annals of Mathematics, ser. 2, vol. 9 (1908), p. 97.

where

$$a_i = \frac{(n-i)!\,i!}{n!}\,p_i, \qquad i = 0, 1, 2, \cdots n.$$

The series of numbers $a_0, a_1, a_2, \cdots a_n$ computed from the coefficients of (1) by means of the relation in (1′) will be referred to as the *binomial coefficients* of the equation (1).

The Fundamental Theorem which we shall now prove may be stated as follows:

If $a_0, a_1, a_2, \cdots a_n$ are the binomial coefficients of an equation of the n^{th} degree; and, if any equation of lower degree whose binomial coefficients are a consecutive set of the given binomial coefficients, e. g.:

$$(2)\quad a_m x^t + t a_{m+1} x^{t-1} + \cdots + \frac{t!}{(t-i)!\,i!} a_{m+i} x^{t-i} + \cdots + t a_{m+t-1} x + a_{m+t} = 0,$$

has imaginary roots; then the original equation has at least as many imaginary roots.

The proof of this theorem follows very easily from two theorems which we shall state as lemmas.

First Lemma. If any derived equation of the given equation (1), i. e.,

$$f^i(x) \equiv \frac{d^i f(x)}{dx^i} = 0,$$

has imaginary roots, the original equation must have at least as many. The truth of this is very easily seen from a figure of the curve $y = f(x)$ by the application of Rolle's Theorem.*

Second Lemma. If any equation has imaginary roots, its reciprocal equation has just as many; and the converse is also true. Since the reciprocal equation is obtained from the original equation by the substitution $x = 1/y$, and its roots are consequently the reciprocals of the roots of the original equation, the truth of this second lemma is seen at once.

The $(n - m - t)^{th}$ derived equation of (1′) can be written down immediately, since the result of the differentiation will be of the $(m + t)^{th}$ degree

* Netto, *Algebra*, vol. 1, p. 208. The theorem is not stated explicitly by Netto in the above form, but follows from his Satz II. The analytical proof is easy.

with corresponding binomial coefficients, after dividing out a factor common to each term of the result;* i. e.,

$$(3)\quad a_0x^{m+t} + (m+t)\,a_1x^{m+t-1} + \cdots + (m+t)\,a_{m+t-1}x + a_{m+t} = 0.$$

The reciprocal equation of (3) is

$$(4)\quad a_{m+t}y^{m+t} + (m+t)\,a_{m+t-1}y^{m+t-1} + \cdots + (m+t)\,a_1y + a_0 = 0.$$

The m^{th} derived equation of (4) is of degree t; and, if we neglect again a certain factor, it can be written:

$$(5)\qquad a_{m+t}y^t + ta_{m+t-1}y^{t-1} + \cdots + ta_{m+1}y + a_m = 0.$$

Finally the reciprocal equation of this last (5) is

$$(6)\qquad a_mx^t + ta_{m+1}x^{t-1} + \cdots + ta_{m+t-1}x + a_{m+t} = 0.$$

Suppose now (6) has $2k$ imaginary roots; then by the second lemma, (5) has $2k$; and in turn by the first lemma, (4) has at least $2k$; then (3) has at least $2k$; and finally by the same reasoning the original equation (1′) has at least $2k$ imaginary roots and hence our fundamental theorem is proved.

Specializations of the Fundamental Theorem. In order to obtain special sufficient conditions for the existence of imaginary roots of (1) or (1′), we have merely to write down any selected number of consecutive binomial coefficients, form the corresponding equation (with binomial factors), and impose the condition that it have imaginary roots. We shall thus obtain a set of relations such that if any one is satisfied the original equation has imaginary roots.

The simplest set of conditions of this kind is that in which three consecutive coefficients are chosen and quadratic equations are formed. For an equation of the n^{th} degree there are $(n-1)$ such quadratics of the type:

$$(7)\qquad a_{i-1}x^2 + 2a_ix + a_{i+1} = 0, \qquad i = 1, 2, 3 \cdots n-1.$$

If now for any one of these equations

$$(8)\qquad a_i^2 - a_{i-1}a_{i+1} < 0,$$

equation (7), and consequently (1′), will have at least one pair of imaginary

* Burnside and Panton, *Theory of Equations*, vol. 1, p. 68. The differentiation of the general term of (1′) with a subsequent slight change in the form of the numerical factor shows this at once.

roots. If we return to the notation of (1) by means of the relation between the a's and p's as given in (1′), we shall have the result:

If any one of the following inequalities is satisfied:

$$(8') \qquad p_i^2 - \frac{i+1}{i}\,\frac{n-i+1}{n-i}\,p_{i-1}p_{i+1} < 0, \qquad i = 1, 2, 3, \cdots n-1,$$

the equation (1) *has at least one pair of imaginary roots.**

This result may also be stated in the following form:

If the equation (1) *has all its roots real, then the following* $n-1$ *inequalities must hold*:

$$(8'') \qquad p_i^2 - \frac{i+1}{i}\,\frac{n-i+1}{n-i}\,p_{i-1}p_{i+1} \geqq 0, \qquad i = 1, 2, 3, \cdots n-1.$$

The condition given by Professor Kellogg† corresponding to (8′) would read: if for any value of i

$$p_i^2 - p_{i-1}p_{i+1} \leqq 0, \qquad i = 1, 2, 3, \cdots n-1,$$

then there are imaginary roots. In the cases in which either test applies, i. e., when p_{i-1} and p_{i+1} have the same sign, it is seen that (8′) gives a somewhat sharper test.

This method of obtaining sufficient conditions for imaginary roots can be extended. We could for example make use of a series of cubics:

$$(9) \qquad a_i x^3 + 3a_{i+1}x^2 + 3a_{i+2}x + a_{i+3} = 0,$$

and, by imposing the condition that these cubics have imaginary roots, we would obtain a series of $n-2$ inequalities such that, if any one is satisfied, the original equation has imaginary roots.

Relation to the Newton-Sylvester Theorem. The condition (8′) is a special case of the theorem of Newton previously referred to. Newton's theorem may be stated as follows:

From the coefficients of the equation (1) the following sequence of $(n+1)$ numbers is formed:

$$(10) \quad p_0^2,\ p_1^2 - l_1 p_0 p_2,\ p_2^2 - l_2 p_1 p_3,\ \cdots\ p_i^2 - l_i p_{i-1} p_{i+1},\ \cdots\ p_n^2,$$

* The case in which three consecutive coefficients are zero will be considered later; and it will be shown that in this case there are imaginary roots.

† Kellogg, loc. cit.

where

$$l_i = \frac{i+1}{i}\,\frac{n-i+1}{n-i}\ ;$$

then the number of real roots of (1) is not greater than the number of permanences of sign in the sequence (10), the difference being even; and therefore the number of imaginary roots is not less than the number of variations of sign in the same sequence. In particular, if any one member of the sequence is negative, the equation has imaginary roots; and this is precisely the result (8′). If all the roots of the equation are real, then the members of (10) must be either positive or zero.

This theorem was given by Newton* without proof and remained without proof until Sylvester† proved a more general theorem, from which Newton's theorem was obtained as a specialization, just as Descartes' Rule of Signs can be obtained as a specialization of the theorem of Budan-Fourier.

Derivation of Other Special Tests. A well known theorem in regard to the roots of an equation with one or more zero coefficients can be easily derived from the Fundamental Theorem.

Suppose in the equation (1)

$$p_{m+1} = p_{m+2} = \cdots = p_{m+t} = 0, \tag{11}$$

while $p_m \neq 0$ and $p_{m+t+1} \neq 0$; then the equation corresponding to (2) takes the form:

$$a_m x^{t+1} + a_{m+t+1} = 0.$$

If t is even, this equation has t imaginary roots; if t is odd, it has $t+1$ or $t-1$ according as a_m and a_{m+t+1} have like or unlike signs, or according as p_m and p_{m+t+1} have like or unlike signs.

Hence, by the Fundamental Theorem, *if any equation have t successive coefficients zero, there are at least t imaginary roots if t is an even number; if t is odd, there are at least $t+1$ or $t-1$ according as the coefficients just before and after those vanishing have like or unlike signs.*‡

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* I. Newton, *Arithmetica universalis*, Cambridge, 1707; second edition, London, 1722, chap. II.

† J. J. Sylvester, *Phil. Mag.* ser. 4, vol. 31 (1866), p. 214. For a proof of this theorem see E. Netto, loc. cit., vol. 1, pp. 225–235.

‡ Cf. Burnside and Panton, *Theory of Equations*, vol. 1, p. 197, where this result is deduced from the Budan-Fourier Theorem; also Netto, *Algebra*, vol. 1, p. 221.

This result can now be applied in turn to the original equation (1) or to any one of the equations (2) derived from (1) in order to obtain other forms of sufficient conditions for imaginary roots. As an example of this method, let us multiply the left-hand side of (1) by the left-hand side of

$$x^r + c_1 x^{r-1} + c_2 x^{r-2} + \cdots + c_{r-1} x + c_r = 0. \tag{12}$$

We shall thus obtain an equation of degree $(n + r)$, which has the same number of imaginary roots as (1) provided that all the roots of (12) are real. Hence if we can choose a set of c's so that all the roots of (12) are real and also so that two successive coefficients of the resulting equation are zero, i. e., so that for a special value of i,

$$\begin{aligned} &c_r p_i + c_{r-1} p_{i+1} + \cdots + c_1 p_{i+r-1} + p_{i+r} = 0, \\ &c_r p_{i+1} + c_{r-1} p_{i+2} + \cdots + c_1 p_{i+r} + p_{i+r+1} = 0; \end{aligned} \tag{13}$$

then the new equation, by the theorem above, has imaginary roots; and consequently the equation (1) must have imaginary roots.

For $r = 1$, all of these conditions are satisfied if

$$\begin{vmatrix} p_i & p_{i+1} \\ p_{i+1} & p_{i+2} \end{vmatrix} = 0;$$

hence:

*If three consecutive coefficients of an equation are in geometric progression, the equation has imaginary roots.**

For $r = 2$, (12) and (13) lead to the result of Professor Kellogg † that if the quadratic

$$\begin{vmatrix} x^2 & x & 1 \\ p_{i+2} & p_{i+1} & p_i \\ p_{i+3} & p_{i+2} & p_{i+1} \end{vmatrix} = 0 \tag{14}$$

has real roots, the equation (1) has imaginary roots.

The special case in which (12) becomes $(x - 1)^2$, and (13) becomes

$$\begin{aligned} p_i - 2p_{i+1} + p_{i+2} &= 0, \\ p_{i+1} - 2p_{i+2} + p_{i+3} &= 0, \end{aligned}$$

* This also follows directly from (8'), which is satisfied whenever the three coefficients are in geometrical progression.

†Kellogg, loc. cit.

gives the theorem of Hermite*: *If four successive coefficients of an equation are in arithmetic progression, the equation has imaginary roots.*

A result analogous to (14) may be obtained in a different way. Consider any one of the equations (2) and replace x by $y + h$ and apply to the result the condition (8); for simplicity, consider the cubic (9) which becomes after the substitution:

$$a_i y^3 + 3\phi_1(h)\, y^2 + 3\phi_2(h)\, y + \phi_3(h) = 0, \tag{15}$$

where

$$\phi_j(h) = a_i h^j + j a_{i+1} h^{j-1} + \frac{j(j-1)}{2!} a_{i+2} h^{j-2} + \cdots.\dagger$$

If we can find a real value of h so that

$$\begin{vmatrix} \phi_2(h) & \phi_1(h) \\ \phi_3(h) & \phi_2(h) \end{vmatrix} < 0,$$

then by (8) the equation (15) has imaginary roots and therefore equation (1′) has also imaginary roots. After a simple reduction this inequality becomes:

$$\begin{vmatrix} a_{i+1}h + a_{i+2} & a_i h + a_{i+1} \\ a_{i+2}h + a_{i+3} & a_{i+1} h + a_{i+2} \end{vmatrix} = \begin{vmatrix} h^2 & -h & 1 \\ a_{i+2} & a_{i+1} & a_i \\ a_{i+3} & a_{i+2} & a_{i+1} \end{vmatrix} < 0. \tag{16}$$

This inequality can be satisfied if the determinant on the left has real and distinct roots, i. e., if

$$\begin{vmatrix} a_{i+2} & a_i \\ a_{i+3} & a_{i+1} \end{vmatrix}^2 - 4 \begin{vmatrix} a_{i+2} & a_{i+1} \\ a_{i+3} & a_{i+2} \end{vmatrix} \cdot \begin{vmatrix} a_{i+1} & a_i \\ a_{i+2} & a_{i+1} \end{vmatrix} > 0. \ddagger \tag{17}$$

If then (17) *is true, the original equation has imaginary roots.*

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* E. Netto, loc. cit., p. 225. This theorem also follows from (14).

† See Burnside and Panton, *Theory of Equations*, vol. 1, p. 68.

‡ This reduces to

$$\frac{1}{a_i^2}\left[\left(a_{i+3}a_i^2 - 3a_i a_{i+1}a_{i+2} + 2a_{i+1}^3\right)^2 + 4\left(a_i a_{i+2} - a_{i+1}^2\right)^3\right] > 0,$$

in which the square brackets enclose the discriminant of the cubic (9). This then is the best test of this kind which it is possible to obtain.

Combination of Van Vleck's Method with the Fundamental Theorem. A method similar to that employed by Professor Van Vleck* can be used to derive from the Fundamental Theorem another form of sufficient condition for imaginary roots.

Given an algebraic equation of degree k, written in the form (1); then if k is even and greater than two, and if

$$(18)\quad q_1^2 - 2q_0\,q_2 < 0, \qquad q_{k-1}^2 - 2q_{k-2}\,q_k < 0, \qquad q_i^2 - q_{i-1}\,q_{i+1} < 0, \quad i = 3, 5, 7, \cdots k-3,$$

all its roots are imaginary.† For from (18) it is clear that none of the coefficients with even subscripts can be zero and also that they must have the same sign.

Writing the equation in the following way:

$$(19)\quad x^{k-2}\left(q_0x^2 + q_1x + \frac{q_2}{2}\right) + \frac{x^{k-4}}{2}\left(q_2x^2 + 2q_3x + q_4\right) + \frac{x^{k-6}}{2}\left(q_4x^2 + 2q_5x + q_6\right) + \cdots + \frac{x^2}{2}\left(q_{k-4}x^2 + 2q_{k-3}x + q_{k-2}\right) + \left(\frac{q_{k-2}}{2}x^2 + q_{k-1}x + q_k\right) = 0,$$

it is seen that the expression can never vanish for real values of x; since from (18) the quadratic expressions in the brackets cannot vanish for real values of x and for such values have the same sign (that of the coefficients of even subscripts).

Let us consider now one of the equations (2) derived from the original equation written with binomial coefficients, supposing t to be even. If we apply the result above to this equation, we shall obtain the following result in regard to the roots of (1′):

If any consecutive set of coefficients of the equation (1′) satisfy the following inequalities:

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* Loc. cit.

†For $k = 2$, replace condition (18) by $q_1^2 - 4q_0q_2 < 0$.

$$a^2_{m+1} - \frac{t-1}{t}\, a_m a_{m+2} < 0,$$

$$(20) \qquad a^2_{m+i} - \frac{i}{i+1}\,\frac{t-i}{t-i+1}\, a_{m+i-1} a_{m+i+1} < 0,$$

$$a^2_{m+t-1} - \frac{t-1}{t}\, a_{m+t} a_{m+t-2} < 0, \qquad i = 3, 5, 7, \cdots t-3,$$

then the equation (1′) has at least t imaginary roots.*

This would not afford a convenient test, owing to the dependence of the numerical coefficients upon the number of inequalities in (20). This difficulty can be removed by a slight sacrifice of sharpness in the test; for in no case will any of these numerical coefficients be so small as $\frac{1}{2}$.† We have then the following result:

If any consecutive set of coefficients of the equation (1′) *satisfy the following inequalities:*

$$(21) \qquad a^2_{m+i} - \tfrac{1}{2}\, a_{m+i-1}\, a_{m+i+1} < 0, \quad i = 1, 3, 5, \cdots t-1,$$

the equation (1′) *has at least t imaginary roots.*

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*For $t = 2$, replace (20) by $a^2_{m+1} - a_m a_{m+2} < 0$.

† In fact they are greater than or equal to 9/16 and this fraction could be used in the inequality (21) instead of the $\frac{1}{2}$.

CONCERNING A COMPOUND DISCONTINUOUS SOLUTION IN THE PROBLEM OF THE SURFACE OF REVOLUTION OF MINIMUM AREA

By Mary E. Sinclair

Introduction. Among the minimum surfaces which can be represented by liquid films is a surface of revolution across which is a transverse plane circular film.* It can be constructed by placing one on the other two equal rings, dipping them into soap solution, and then drawing them apart always parallel to each other and perpendicular to an axis through their centers. A longitudinal section of the surface shows on each side of this axis a system of three curves meeting in a point, two of the curves extending respectively to the two rings, and the third extending to the axis. The experiment suggests the following mathematical problem:

Given two points A_0 and A_1 in the xy-plane and on the same side of the x-axis. Join the points by a curve confined to the positive side of the x-axis, and from an arbitrary point P_2 of this curve draw a curve to the x-axis. To find among all such systems of curves that one which when revolved about the x-axis shall determine a surface of revolution of minimum area. All curves which we consider will be of class D'.†

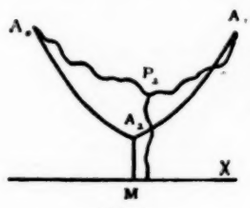

FIG. 1.

The object of the present paper is to give a solution of this problem. The discussion is divided into three parts A, B, C, dealing, respectively, with the necessary conditions for the solution, the sufficient conditions, and the character of the field of extremals. The principal results obtained are as follows:

1. The solution must consist of two catenaries $A_0 A_2$, $A_2 A_1$, with the x-axis for directrix, and the normal $A_2 M$ from A_2 to the x-axis. The catenaries must meet the normal $A_2 M$ under an angle of 120°. (A, §§ 1–3.)

2. On the arc $A_0 A_2$ there exists a critical point A_0', a limit on the left

* See Plateau, *Statiques des liquides*, for a general discussion of minimum surfaces which can be represented by liquid films

† Bolza, *Lectures on the Calculus of Variations*, §2. A curve is of class D' if it is continuous and consists of a finite number of arcs each of which has a continuously turning tangent.

for the position of A_0: on the arc $A_2 A_1$ there exists a conjugate point A_0', a limit on the right for the position of A_1. For the latter point there exists a simple geometric construction which we shall call the Lindelöf construction for our problem. (A, §§ 4–9.)

3. The above are not only necessary but also sufficient conditions for a relative minimum (B).

4. The solution is unique within a field defined by the one-parameter set of extremal-systems passing through A_0, defined in 1 above. (C, §§ 1–3.)

5. The field consists of two parts in which the area of the surface of revolution given by our solution is respectively greater or smaller than that given by the solution found by Goldschmidt.* (C, §4.)

6. Experiment verifies the results defining the position of the conjugate point.† (C, §5.)

A. Necessary Conditions for the Solution.

1. The system consisting of segments of two catenaries and one straight line. We suppose a totality $\mathfrak{M}$ of systems of three curves of the form $x = x(t)$, $y = y(t)$, which, as indicated above,

1) are of class D',

2) lie in the region $y > 0$,

3) meet in a point P_2, joining it respectively to the given points A_0 and A_1, and the x-axis.

We suppose further that a particular system through a point A_2, of which we shall denote the segment $A_0 A_2$ by $\mathfrak{E}_0$, the segment $A_2 A_1$ by $\mathfrak{E}_1$, and the segment joining A_2 to the x-axis by $\mathfrak{E}_2$, actually furnishes a relative minimum for the integral

$$I = \int_{\mathfrak{E}_0} 2\pi y \sqrt{x'^2 + y'^2}\, dt + \int_{\mathfrak{E}_1} 2\pi y \sqrt{x'^2 + y'^2}\, dt + \int_{\mathfrak{E}_2} 2\pi y \sqrt{x'^2 + y'^2}\, dt,$$

where x' and y' are the derivatives dx/dt and dy/dt.

* Goldschmidt, *Prize Essay*, 1830.

† After the completion of this paper, two papers by H. Tallqvist dealing with the same problem came to my attention: "Determination experimentale de la limite de stabilité de quelques surfaces minima," *Transactions of the Scientific Society of Finland*, vol. 32 (1889); and *Bestimmung einiger Minimalflächen deren Begrenzung gegeben ist*, Helsingfors, 1890. Tallqvist's results concerning the limit of stability, obtained by an entirely different method, agree with our own results in the special case of equal rings; they are wrong in the case of unequal rings, as will be shown in C, §5, which I have added for the sake of comparison.

We vary each of the segments $\mathfrak{E}_0$, $\mathfrak{E}_1$, $\mathfrak{E}_2$ separately, leaving the remainder of the system fixed. By the usual theory, each of these curves must in the first place be a solution of the Euler equation

$$F_{xy'} - F_{xy} + F_1 (x'y'' - x''y') = 0,$$

where

$$F = 2\pi y \sqrt{x'^2 + y'^2},$$

and therefore for our problem must be either a catenary,*

$$x = at + \beta, \qquad y = a \operatorname{ch} t,$$

or a vertical straight line,

$$x = a, \qquad y = t.$$

In particular $\mathfrak{E}_0$ and $\mathfrak{E}_1$ must be catenaries, and $\mathfrak{E}_2$ must be a straight line, since no catenary of the set given above touches the x-axis. The points A_i have coordinates (a_i, b_i), and we write

$$(1)\quad \begin{aligned} \mathfrak{E}_0 &: \begin{cases} x = a_0 t + \beta_0, \\ y = a_0 \operatorname{ch} t, \end{cases} \text{ where } \begin{cases} a_0 = a_0 t_0 + \beta_0, \\ b_0 = a_0 \operatorname{ch} t_0, \end{cases} \text{ and } \begin{cases} a_2 = a_0 t_2 + \beta_0, \\ b_2 = a_0 \operatorname{ch} t_2; \end{cases} \\ \mathfrak{E}_1 &: \begin{cases} x = a_1 \tau + \beta_1, \\ y = a_1 \operatorname{ch} \tau, \end{cases} \text{ where } \begin{cases} a_2 = a_1 \tau_2 + \beta_1, \\ b_2 = a_1 \operatorname{ch} \tau_2, \end{cases} \text{ and } \begin{cases} a_1 = a_1 \tau_1 + \beta_1, \\ b_1 = a_1 \operatorname{ch} \tau_1; \end{cases} \\ \mathfrak{E}_2 &: \quad x = a_2. \end{aligned}$$

Further, $\mathfrak{E}_0$, $\mathfrak{E}_1$, $\mathfrak{E}_2$ must each satisfy the conditions numbered by Bolza I, II, III, and IV,† which are necessary for a minimum. The stronger forms II′ and IV′ are always satisfied‡ since along the system of curves under consideration

$$F_1(xy \cos\gamma \sin\gamma) = \frac{F_{x'x'}}{y'^2} = y > 0.$$

We may also immediately require that condition III be satisfied in the stronger form III′, by the arcs $\mathfrak{E}_0$ and $\mathfrak{E}_1$, since the envelope of the one-parameter set of catenaries through A_2 has no singular point, and therefore

* Bolza, loc. cit., p. 153.

† Loc. cit., §25–28.

‡ Bolza, loc. cit., p. 146.

the arc $\mathfrak{E}_0$ from A_2 to its conjugate A_2' could never furnish a minimum.* A similar remark holds for $\mathfrak{E}_1$. *The two catenaries and the normal to the x-axis, which form the system, must therefore separately satisfy all the conditions for fixed end points.*

2. The integral as a function of the coordinates of P_2, the corner point. We shall apply the following theorem:†

Let $M_0(t = t_0,\ x = a_0,\ y = b_0)$ and $M_1(t = t_1,\ x = a_1,\ y = b_1)$ be two non-conjugate points on any extremal. Then it is always possible to construct through the points $P_0(x = x_0,\ y = y_0)$ and $P_1(x = x_1,\ y = y_1)$ in the neighborhood of M_0 and M_1 respectively, a unique extremal, over which the integral is a single-valued, continuous function of the coordinates of P_0 and P_1,

$$J(x_0,\ y_0,\ x_1,\ y_1),$$

with continuous partial derivatives in the vicinity of $(a_0,\ b_0,\ a_1,\ b_1)$. Further, the total differential of J is ‡

$$\begin{aligned} dJ(x_0,\ y_0,\ x_1,\ y_1) = {} & F_{x'}(x_1,\ y_1,\ x_1',\ y_1')\ dx_1 + F_{y'}(x_1,\ y_1,\ x_1',\ y_1',)dy_1 \\ & - F_{x'}(x_0,\ y_0,\ x_0',\ y_0')\ dx_0 - F_{y'}(x_0,\ y_0,\ x_0',\ y_0')dy_0. \end{aligned}$$

For our problem, therefore,

$$(2)\quad dJ(x_0,\ y_0,\ x_1,\ y_1)$$
$$= 2\pi\left\{\frac{y_1x_1'}{\sqrt{x_1'^2 + y_1'^2}}dx_1 + \frac{y_1y_1'}{\sqrt{x_1'^2 + y_1'^2}}dy_1 - \frac{y_0x_0'}{\sqrt{x_0'^2 + y_0'^2}}dx_0 - \frac{y_0y_0'}{\sqrt{x_0'^2 + y_0'^2}}dy_0\right\}.$$

For the extremal-system, $\mathfrak{E}_0$, $\mathfrak{E}_1$, $\mathfrak{E}_2$, the integral I is completely determined in terms of this function J, its value being

$$(3)\qquad I(\mathfrak{E}_0,\ \mathfrak{E}_1,\ \mathfrak{E}_2) = J(a_0,\ b_0,\ a_2,\ b_2) + J(a_2,\ b_2,\ a_1,\ b_1) + \pi\, b_2^2.$$

Suppose $P_2(x_2,\ y_2)$ any point in the vicinity of A_2. Then if the catenary through A_0 and P_2 be $\bar{\mathfrak{E}}_0$, and that through P_2 and A_1 be $\bar{\mathfrak{E}}_1$, we have

---

* Bolza, loc. cit., p. 204.

† Bolza, loc. cit., p. 174, footnote 1.

‡ Bolza, loc. cit., p. 176.

$$(4)\qquad \begin{aligned} \bar{\mathfrak{E}}_0&: \quad x = \bar{a}_0\bar{t} + \bar{\beta}_0, \quad y = \bar{a}_0 \operatorname{ch}\bar{t}; \\ \bar{\mathfrak{E}}_1&: \quad x = \bar{a}_1\bar{\tau} + \bar{\beta}_1, \quad y = \bar{a}_1 \operatorname{ch}\bar{\tau}; \\ \bar{\mathfrak{E}}_2&: \quad x = x_2, \end{aligned}$$

with equations similar to those given under (1) at the end points. Over this system, the integral I is a function of x_2, y_2, namely:

$$(5)\quad I(\bar{\mathfrak{E}}_0, \bar{\mathfrak{E}}_1, \bar{\mathfrak{E}}_2) = \phi(x_2, y_2) = J(a_0, b_0, x_2, y_2) + J(x_2, y_2, a_1, b_1) + \pi y_2^2.$$

It is now necessary that the function $\phi(x_2, y_2)$ have a minimum for the values $x_2 = a_2$, $y_2 = b_2$. By the ordinary theory of maxima and minima, the following conditions must be satisfied for $x_2 = a_2$, $y_2 = b_2$:

$$(6)\quad \frac{\partial\phi}{\partial x_2} = 0, \quad \frac{\partial\phi}{\partial y_2} = 0, \quad \frac{\partial^2\phi}{\partial x_2\partial x_2} \gtreqless 0, \quad \left(\frac{\partial^2\phi}{\partial x_2\partial y_2}\right)^2 - \left(\frac{\partial^2\phi}{\partial x_2\partial x_2}\right)\left(\frac{\partial^2\phi}{\partial y_2\partial y_2}\right) \leqq 0.$$

3. The corner condition. We first discuss the conditions on the first derivatives of ϕ with respect to x_2 and y_2, where $x_2 = a_2$ and $y_2 = b_2$. From (2) and (5), we have

$$\frac{\partial\phi}{\partial x_2} = 2\pi\frac{y_2x_2'}{\sqrt{x_2'^2 + y_2'^2}} - 2\pi\frac{y_2\tilde{x}_2'}{\sqrt{\tilde{x}_2'^2 + \tilde{y}_2'^2}},$$

$$\frac{\partial\phi}{\partial y_2} = 2\pi\frac{y_2y_2'}{\sqrt{x_2'^2 + y_2'^2}} - 2\pi\frac{y_2\tilde{y}_2'}{\sqrt{\tilde{x}_2'^2 + \tilde{y}_2'^2}} + 2\pi y_2,$$

where x_2', y_2' refer to $\bar{\mathfrak{E}}_0$ and $\tilde{x}_2'$, $\tilde{y}_2'$ refer to $\bar{\mathfrak{E}}_1$. From (4), we see that for $x_2 = a_2$ and $y_2 = b_2$,

$$x_2' = a_0, \quad y_2' = a_0 \operatorname{sh} t_2, \quad \sqrt{x_2'^2 + y_2'^2} = y_2,$$

and

$$\tilde{x}_2' = a_1, \quad \tilde{y}_2' = a_1 \operatorname{sh}\tau_2, \quad \sqrt{\tilde{x}_2'^2 + \tilde{y}_2'^2} = y_2.$$

Hence at the point (a_2, b_2) we must have

$$(7)\qquad \frac{\partial\phi}{\partial x_2} = 2\pi(a_0 - a_1) = 0,$$

and hence

$$a_0 = a_1. \tag{8}$$

Moreover, (6b) requires

$$\frac{\partial \phi}{\partial y_2} = 2\pi a_0(\operatorname{sh} t_2 - \operatorname{sh} \tau_2 + \operatorname{ch} t_2) = 0. \tag{9}$$

But

$$b_2 = a_0 \operatorname{ch} t_2 = a_1 \operatorname{ch} \tau_2,$$

and therefore

$$t_2 = \pm \tau_2.$$

Using the upper sign, (9) becomes impossible, since $a_0 \neq 0$ and $\operatorname{ch} t_2 \neq 0$. Hence

$$t_2 = -\tau_2, \tag{10}$$

and the equations

$$a_2 = a_0 t_2 + \beta_0 = a_1 \tau_2 + \beta_1$$

give

$$a_2 = \tfrac{1}{2}(\beta_0 + \beta_1). \tag{11}$$

Condition (9) now requires

$$2 + \coth t_2 = 0;$$

from which, if we represent by p_2 the slope of $\mathfrak{E}_0$ at A_2 and by π_2 that of $\mathfrak{E}_1$ at A_2, we have

$$\begin{aligned} t_2 &= -\log \sqrt{3}, & p_2 &= \operatorname{sh} t_2 = -\frac{1}{\sqrt{3}}, \\ \tau_2 &= \log \sqrt{3}, & \pi_2 &= \operatorname{sh} \tau_2 = +\frac{1}{\sqrt{3}}. \end{aligned} \tag{12}$$

We obtain from equations (8) and (12), therefore, the result that *the vertices of* $\mathfrak{E}_0$ *and* $\mathfrak{E}_1$ *must be at the same distance from the* x*-axis, and that the three curves* $\mathfrak{E}_0$, $\mathfrak{E}_1$, $\mathfrak{E}_2$ *must all meet at an angle of* 120°.*

4. Critical points. We shall now discuss the remaining conditions of (6).

* Physically, a result of equal tension; Tallqvist, *Determination experimentale* . . ., loc. cit., p. 7. Compare also Plateau, loc. cit., vol. 1, §173, p. 294, where the same result appears for a system of two equal spherical surfaces and their plane of intersection.

We obtain partial derivatives of $\bar{a}_0$, $\bar{a}_1$, $\bar{t}_2$, $\bar{\tau}_2$ with respect to x_2 and y_2 from equations for the end points of $\mathfrak{E}_0\,\mathfrak{E}_1$ similar to those given under (1). Evidently

$$
\begin{aligned}
x_2 - a_0 &= \bar{a}_0(\bar{t}_2 - \bar{t}_0), \\
b_0 &= \bar{a}_0 \operatorname{ch}\bar{t}_0, \\
y_2 &= \bar{a}_0 \operatorname{ch}\bar{t}_2.
\end{aligned} \tag{13}
$$

Differentiating (13) with respect to x_2, and putting $x_2 = a_2$, $y_2 = b_2$, we obtain

$$
\begin{aligned}
1 &= \frac{\partial a_0}{\partial x_2}(t_2 - t_0) + a_0\left(\frac{\partial t_2}{\partial x_2} - \frac{\partial t_0}{\partial x_2}\right), \\
0 &= \frac{\partial a_0}{\partial x_2}\operatorname{ch} t_0 + a_0 \operatorname{sh} t_0 \frac{\partial t_0}{\partial x_2}, \\
0 &= \frac{\partial a_0}{\partial x_2}\operatorname{ch} t_2 + a_0 \operatorname{sh} t_2 \frac{\partial t_2}{\partial x_2};
\end{aligned}
$$

whence,

$$
\frac{\partial a_0}{\partial x_2} = \frac{1}{t_2 - t_0 - \coth t_2 + \coth t_0}.
$$

We shall use the abbreviation $\chi(t) = \coth t - t$.
Then

$$
\frac{\partial a_0}{\partial x_2} = \frac{1}{\chi(t_0) - \chi(t_2)}. \tag{14}
$$

Differentiating (13) with respect to y_2, and putting $x_2 = a_2$, $y_2 = b_2$, we obtain

$$
\begin{aligned}
0 &= \frac{\partial a_0}{\partial y_2}(t_2 - t_0) + a_0\left(\frac{\partial t_2}{\partial y_2} - \frac{\partial t_0}{\partial y_2}\right), \\
0 &= \frac{\partial a_0}{\partial y_2}\operatorname{ch} t_0 + a_0 \operatorname{sh} t_0 \frac{\partial t_0}{\partial y_2}, \\
1 &= \frac{\partial a_0}{\partial y_2}\operatorname{ch} t_2 + a_0 \operatorname{sh} t_2 \frac{\partial t_2}{\partial y_2};
\end{aligned}
$$

whence

$$
\frac{\partial a_0}{\partial y_2} = -\frac{1}{\chi(t_0) - \chi(t_2)} \cdot \frac{1}{\operatorname{sh} t_2} = -\frac{1}{\operatorname{sh} t_2}\frac{\partial a_0}{\partial \chi_2}. \tag{15}
$$

Similarly, we obtain

$$\frac{\partial a_1}{\partial x_2} = \frac{1}{\chi(\tau_1) - \chi(\tau_2)},$$

$$\frac{\partial a_1}{\partial y_2} = -\frac{1}{\operatorname{sh}\tau_2} \cdot \frac{\partial a_1}{\partial x_2}. \tag{16}$$

Further,

$$\frac{\partial t_2}{\partial y_2} = \frac{1}{a_0 \operatorname{sh} t_2}\left[1 + \frac{\operatorname{ch} t_2}{\operatorname{sh} t_2} \cdot \frac{\partial a_0}{\partial x_2}\right],$$

$$\frac{\partial \tau_2}{\partial y_2} = \frac{1}{a_1 \operatorname{sh} \tau_2}\left[1 + \frac{\operatorname{ch} \tau_2}{\operatorname{sh} \tau_2} \cdot \frac{\partial a_1}{\partial x_2}\right]. \tag{17}$$

Making use of (15), (16), (17), we now obtain the second derivatives at the point (a_2, b_2):

$$\frac{\partial^2 \phi}{\partial x_2 \partial x_2} = 2\pi\left(\frac{\partial a_0}{\partial x_2} - \frac{\partial a_1}{\partial x_2}\right),$$

$$\frac{\partial^2 \phi}{\partial x_2 \partial y_2} = 2\pi\left(-\frac{1}{\operatorname{sh} t_2}\frac{\partial a_0}{\partial x_2} + \frac{1}{\operatorname{sh}\tau_2}\frac{\partial a_1}{\partial x_2}\right), \tag{18}$$

$$\frac{\partial^2 \phi}{\partial y_2 \partial y_2} = 2\pi\left(1 + \coth t_2 - \coth \tau_2 + \frac{1}{\operatorname{sh} t_2}\frac{\partial a_0}{\partial x_2} - \frac{1}{\operatorname{sh}\tau_2}\frac{\partial a_1}{\partial x_2}\right).$$

Condition (6c) therefore requires

$$\left(\frac{\partial a_0}{\partial x_2} - \frac{\partial a_1}{\partial x_2}\right) = \frac{1}{\chi(t_0) - \chi(t_2)} + \frac{1}{\chi(\tau_2) - \chi(\tau_1)} \gtrless 0. \tag{19}$$

But $\chi(t)$ is a decreasing function, continuous except at $t = 0$. Hence, since $t_0 < t_2 < 0$ and $0 < \tau_2 < \tau_1$, each fraction is positive, and *condition* (6c) *is always fulfilled in the stronger form*

$$\frac{\partial^2 J}{\partial x_2 \partial x_2}\bigg|^{x_2 = a_0} > 0.$$

Since $-\operatorname{sh} t_2 = \operatorname{sh}\tau_2 = 1/\sqrt{3}$, condition (6d) requires that for $x_2 = a_2$, $y_2 = b_2$,

$$3\left(\frac{\partial a_0}{\partial x_2} + \frac{\partial a_1}{\partial x_2}\right)^2 - \left(\frac{\partial a_0}{\partial x_2} - \frac{\partial a_1}{\partial x_2}\right)\left[-3 + 3\left(\frac{\partial a_0}{\partial x_2} - \frac{\partial a_1}{\partial x_2}\right)\right] \gtrless 0,$$

or

$$4\,\frac{\partial a_0}{\partial x_2}\cdot\frac{\partial a_1}{\partial x_2}+\frac{\partial a_0}{\partial x_2}-\frac{\partial a_1}{\partial x_2}\gtreqless 0.$$

If we apply (15) and (16), this becomes

$$\frac{\log 3+\chi(\tau_1)-\chi(t_0)}{\{\chi(\tau_1)-\chi(\tau_2)\}\{\chi(t_0)-\chi(t_2)\}}\gtreqless 0.$$

Since the denominator is always negative, this condition is equivalent to the following:

$$\log 3+\chi(\tau_1)-\chi(t_0)\geqq 0, \tag{20}$$

from which two results may be derived.

Let t_0 have any fixed value, and let τ_1 increase from τ_2 on. Then since $\chi(t)$ is a decreasing function, $\log 3+\chi(\tau_1)-\chi(t_0)$ decreases from $\log 3+\chi(\tau_2)-\chi(t_0)$. Hence, if the latter value is negative, it is impossible to satisfy (20). But

$$\chi(\tau_2)=2-\log\sqrt{3},$$

and hence

$$\log 3+\chi(\tau_2)-\chi(t_0)=2+\log\sqrt{3}-\chi(t_0).$$

It is therefore necessary that

$$\chi(t_0)\leqq 2+\log\sqrt{3}. \tag{21}$$

If we denote by t_0' the negative value of t satisfying the equation,

$$\chi(t)=2+\log\sqrt{3}, \tag{22}$$

it is then necessary that

$$t_0\leqq t_0'. \tag{23}$$

The above condition (22) being satisfied, condition (20) further requires that τ_1 satisfy the condition

$$\chi(\tau_1)\geqq -\log 3+\chi(t_0). \tag{24}$$

If we denote by t_0'' the positive value of t satisfying the equation

$$\chi(t)=-\log 3+\chi(t_0), \tag{25}$$

it is then necessary that

$$\tau_1\geqq t_0''. \tag{26}$$

For the extreme value $t_0 = t_0'$, t_0' must be equal to τ_2. For the extreme value $t_0 = t_2$, t_0' takes the value t_1' which is the positive value of τ satisfying the equation

$$\chi(\tau) = \chi(t_2) - \log 3 = -2 - \log \sqrt{3}.$$

As t_0 increases from t_0' to t_2, t_0' increases from τ_2 to τ_1'.

5. Geometric construction for critical point. For these conditions restricting the position of A_0 and A_1 we have the following simple geometric construction:

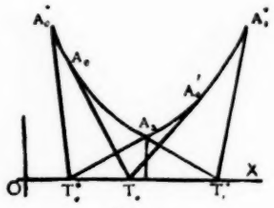

FIG. 2.

Let OT_0 be the x-intercept of the tangent to $\mathfrak{E}_0$ at A_0. Then

$$OT_0 = \beta_0 + at_0 - a \coth t_0.$$

Let OT be the x-intercept of the tangent to $\mathfrak{E}_1$ at A_1. Then

$$OT = \beta_1 + a\tau - a \coth \tau.$$

Hence

$$\begin{aligned} OT_0 - OT &= \beta_0 - \beta_1 + a[t_0 - \tau - \coth t_0 + \coth \tau] \\ &= a(\tau_2 - t_2) + a[\chi(\tau) - \chi(t_0)] \\ &= a[\log 3 + \chi(\tau) - \chi(t_0)]. \end{aligned}$$

The point T coincides with T_0 if and only if

$$\log 3 + \chi(\tau) - \chi(t_0) = 0. \tag{27}$$

Hence the point A_0' ($\tau = t_0'$) is the point on $\mathfrak{E}_1$ determined by the tangent from T_0. We may say that A_0' is the conjugate to A_0 with respect to the broken-extremals $\mathfrak{E}_0\mathfrak{E}_1$, and may call our construction *the Lindelöf construction** for the discontinuous solution.

* Lindelöf, *Mathematische Annalen*, vol. 2 (1870), p. 160.

Let, further, the tangent to $\mathfrak{E}_1$ at A_2 cut the x-axis in T_0^*, and let A_0^* be the point of $\mathfrak{E}_0$ determined by the tangent from T_0^*. Similarly the tangent to $\mathfrak{E}_0$ at A_2 cuts the x-axis in a point T_1^*, and A_1^* is the point of $\mathfrak{E}_1$ determined by the tangent from T_1^*. As t_0 varies from t_0' to t_2, A_0 describes the curve $\mathfrak{E}_0$ from A_0^* to A_2, and at the same time A_0' describes $\mathfrak{E}_1$ from A_2 to A_1^*.

6. The envelope of the broken-extremals. Let us consider the one-parameter family of broken-extremals $(\mathfrak{E}_0\mathfrak{E}_1)$,

$$(28)\qquad \begin{aligned} &\mathfrak{E}_0: \quad x = a_0 + a(t - t_0), \quad y = a \operatorname{ch} t, \quad t_0 \leqq t \leqq t_2, \\ &\mathfrak{E}_1 \quad x = x_2 + a(\tau + t_2), \quad y = a \operatorname{ch} \tau, \quad t_2 \leqq \tau \leqq t_0', \end{aligned}$$

with a as the variable parameter, each of which passes through the point A_0 and satisfies the condition

$$\operatorname{sh} t_2 = \frac{-1}{\sqrt{3}}$$

at the corner point $P_2\ (x_2, y_2)$, where $t = t_2$ and $\tau = -t_2$. The value t_0 is determined by the condition

$$b_0 = a \operatorname{ch} t_0,$$

and we suppose that

$$t_0^* < t_0 \leqq -\log \sqrt{3},$$

so that A_0 is never coincident with the point A_0^* on $\mathfrak{E}_0$.

From the above equations we obtain for the coordinates of P_2:

$$(29)\qquad K: \quad \begin{aligned} x_2 &= a_0 + a(-\log \sqrt{3} - t_0), \\ y_2 &= \frac{2}{\sqrt{3}} a, \qquad \text{where} \qquad b_0 = a \operatorname{ch} t_0. \end{aligned}$$

Since x_2 and y_2 are continuous functions of a, P_2 describes a continuous curve K, which we shall call *the corner curve*.

The set of catenaries $\mathfrak{E}_1$ possesses an envelope, E, given by the second equation of (28) together with the following condition on their functional determinant:

$$\frac{\partial(xy)}{\partial(a\tau)} = 0.$$

The latter gives

$$(30)\qquad \frac{\partial(xy)}{\partial(a\tau)} = \begin{vmatrix} a & a\,\mathrm{sh}\,\tau \\ \tau - t_0 - \log 3 - \dfrac{y_0}{\sqrt{y_0^2 - a^2}} & \mathrm{ch}\,\tau \end{vmatrix}$$

$$= a\,\mathrm{sh}\,\tau\left\{\coth\tau - \tau + \log 3 - t_0 + \frac{y_0}{\sqrt{y_0^2 - a^2}}\right\}$$

$$= a\,\mathrm{sh}\,\tau\{\chi(\tau) - \chi(t_0) + \log 3\} = 0.$$

It therefore follows for a particular system of the set $\mathfrak{E}_0\mathfrak{E}_1\mathfrak{E}_2$ that $A_0'(\tau = t_0')$ *is the point in which the particular catenary* $\mathfrak{E}_1$ *touches the envelope of the set* $\mathfrak{E}_1$. *We shall call this also the envelope of the set of broken-extremals* ($\mathfrak{E}_0$ $\mathfrak{E}_1$) *through* A_0.

7. The physical illustration of the critical point. A physical illustration is now of interest. If we construct a film between two rings as indicated in the introduction, the film is stable so long as the rings are sufficiently near together. At a fixed distance, however, the film becomes unstable, and divides into two separate bubbles which contract into plane circular films.* This point of instability is indeed the conjugate point.

We way readily obtain numerical values from our formulas.

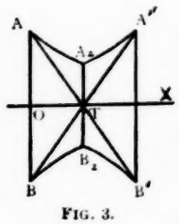

FIG. 3.

Consider the longitudinal section of the film, which for equal rings is symmetrical with respect to the point of intersection of the axis and the transverse film. Let its axis coincide with the x-axis, and the diameter AB of one ring with the y-axis. Let $A'B'$ be the diameter of the second ring and A_2B_2 that of the transverse film. Let $(0, 1)$ be the point A_0 and (x_2, y_2) the point A_2. Then $(2x_2, 1)$ is the point A'. The tangents to the arcs AA_2 and A_2A' at A and A' respectively meet at $T(x_2, 0)$. For the catenary AA_2,

$$\mathfrak{E}_0: \qquad \begin{aligned} y &= a\,\mathrm{ch}\,t, \\ b_0 = 1 &= a\,\mathrm{ch}\,t_0, \\ x &= a(t - t_0). \end{aligned}$$

* The Goldschmidt discontinuous solution.

The coordinates of A_2 are therefore given by the equations

$$A_2: \quad \begin{aligned} x_2 &= a(t_2 - t_0) = -\frac{b_0}{b_0'} = -a\,\frac{\operatorname{ch} t_0}{\operatorname{sh} t_0}, \\ y_2 &= a \operatorname{ch} t_2, \\ t_2 &= -\log\sqrt{3}, \qquad \text{where} \qquad 1 = a \operatorname{ch} t_0. \end{aligned}$$

Hence $$\coth t_0 - t_0 = \log\sqrt{3} = .5493$$

and $$t_0 = -1.6292.$$

It follows that

$$\begin{aligned} a &= \frac{1}{\operatorname{ch} t_0} = .3777, \\ x_2 &= a\,(t_2 - t_0) = .4078, \\ y_2 &= a \operatorname{ch} t_2 = .4343. \end{aligned}$$

From a series of four actual experiments the following coordinates for P_2 were obtained (see also part C, §5, below):

	b_0 ($cm.$)	y_2 ($cm.$)	x_2 ($cm.$)
	3.80	1.700	1.525
	3.80	1.625	1.525
	3.80	1.575	1.525
	3.80	1.625	1.525
average	3.80	1.630	1.525
Reduced to scale $b_0 = 1$	1	.4284	.4013
Theoretical value	1	.4360	.4078

8. Necessity for a stronger form of conditions (26) and (23). For the envelope, E, of the set $\mathfrak{E}_1$, the slope is $dy/dx = \operatorname{sh}\tau > 0$. Hence E has no singular point, and the stronger form of condition (26),

$$\tau_1 < t_0',$$

is necessary.* If $t_0 = t_0'$, τ_2 must be equal to t_1'. Hence the stronger form, $t_0 > t_0'$, of (23) is necessary.

* Bolza, loc. cit., §38, p. 204.

9. Summary of the necessary conditions. The minimizing system of curves must be composed of arcs of two catenaries $\mathfrak{E}_0$, $\mathfrak{E}_1$ having the x-axis as directrix, together with the normal $\mathfrak{E}_2$ to the x-axis from their point of intersection A_2.

$\mathfrak{E}_0$ and $\mathfrak{E}_1$ must be symmetrically situated with respect to $\mathfrak{E}_2$ in such a way that the three curves make angles of 120° with each other, as indicated by equations (8) and (12). The value of t_0 must be greater than t_0', where t_0' is the negative value of t satisfying

$$\chi(t) = 2 + \log \sqrt{3},$$

or in other words, A_0 must lie between A_2 and its conjugate (in sense of §5 above) on the curve $\mathfrak{E}_0$. Furthermore A_1 must lie between A_2 and a point A_0' determined on $\mathfrak{E}_1$ by the equation

$$\chi(\tau_0') = \chi(t_0) - \log \sqrt{3}.$$

The point A_0' is said to be the conjugate point to A_0 on the broken extremal $(\mathfrak{E}_0, \mathfrak{E}_1)$.*

B. Sufficient Conditions.

We now suppose that for the extremal system $\mathfrak{E}_0\,\mathfrak{E}_1\,\mathfrak{E}_2$ the conditions described in the last section are all satisfied. Since the problem is regular and since $t_0 < t_2 < 0$ and $0 < \tau_2 < \tau_1$, it follows that Bolza's conditions I, II′, III′, IV′ are all satisfied for $\mathfrak{E}_0\,\mathfrak{E}_1\,\mathfrak{E}_2$ separately. Further, under the above conditions, a certain vicinity of A_2 $(a_2\,b_2)$,

$$|x_2 - a_2| < \delta, \quad |y_2 - b_2| < \delta,$$

can be assigned in which

$$\phi(a_2\,b_2) < \phi(x_2\,y_2),$$

where $\phi(x_2\,y_2)$ is the function defined in (5) above.

Lemma. The broken-extremal $\mathfrak{E}_0\,\mathfrak{E}_1$ is contained wholly within the region R_0 bounded by the ordinate through A_0 and the envelope F_0 of the set of extremals through A_0. The infinite segment of $\mathfrak{E}_0$ with initial point A_0 lies in R_0. Hence A_2 lies in R_0. Let A_3 be a point on $\mathfrak{E}_0$ symmetrical to A_2 with respect to the axis of $\mathfrak{E}_0$. Then the segment of $\mathfrak{E}_1$ with A_2 as initial point can be obtained by a lateral displacement of the arc $A_3\,\infty$ of $\mathfrak{E}_1$, bringing A_3 into coincidence with A_2', and $\mathfrak{E}_1$ is thus wholly within R_0.

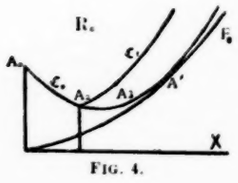

FIG. 4.

* The conditions denoted II′, III′, IV′ by Bolza, are all satisfied for $\mathfrak{E}_0$ and $\mathfrak{E}_1$ separately under the above conditions.

We now construct the envelope F_0 of the set of catenaries through A_0, and the envelope F_1 of the set through A_1, and let S be the region common to the regions R_0 and R_1 which they define. Then A_2 lies in S, and $\mathfrak{E}_0$ and $\mathfrak{E}_1$ lie in S.

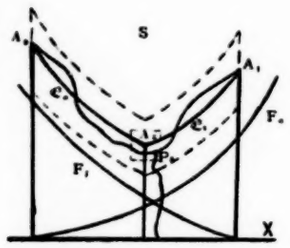

FIG. 5.

We can now take δ so small that the vicinity (δ) lies entirely in S and we can construct a neighborhood (ρ) of $\mathfrak{E}_0$ and $\mathfrak{E}_1$ also lying wholly in S. Let $P_2(x_2, y_2)$, be any point common to (δ) and (ρ) and let $\bar{\mathfrak{E}}_0$, $\bar{\mathfrak{E}}_1$ be catenaries joining P_2 with A_0 and A_1 respectively, and $\mathfrak{C}_0$ and $\mathfrak{C}_1$ any other ordinary curves in (ρ) joining P_2 with A_0 and A_1 respectively. Let $\bar{\mathfrak{E}}_2$ be the normal from P_2 to the x-axis, and $\mathfrak{C}_2$ any ordinary curve from P_2 to the x-axis. Then

$$I(\mathfrak{E}_0, \mathfrak{E}_1, \mathfrak{E}_2) = \phi(a_2, b_2),$$

$$I(\bar{\mathfrak{E}}_0, \bar{\mathfrak{E}}_1, \bar{\mathfrak{E}}_2) = \phi(x_2, y_2),$$

and hence

$$I(\mathfrak{E}_0, \mathfrak{E}_1, \mathfrak{E}_2) < I(\bar{\mathfrak{E}}_0, \bar{\mathfrak{E}}_1, \bar{\mathfrak{E}}_2).$$

But $\quad I(\bar{\mathfrak{E}}_0) < I(\mathfrak{C}_0), \qquad I(\bar{\mathfrak{E}}_1) < I(\mathfrak{C}_1), \qquad I(\bar{\mathfrak{E}}_2) < I(\mathfrak{C}_2),$

and therefore

$$I(\bar{\mathfrak{E}}_0, \bar{\mathfrak{E}}_1, \bar{\mathfrak{E}}_2) < I(\mathfrak{C}_0, \mathfrak{C}_1, \mathfrak{C}_2).$$

Hence we have

$$I(\mathfrak{E}_0, \mathfrak{E}_1, \mathfrak{E}_2) < I(\mathfrak{C}_0, \mathfrak{C}_1, \mathfrak{C}_2).$$

*The conditions given in A, §9 are therefore not only necessary but sufficient conditions that the extremal-system $\mathfrak{E}_0\,\mathfrak{E}_1\,\mathfrak{E}_2$ furnish a relative minimum with respect to the admissible curves defined in A, §1, above.**

* Our problem is similar to those considered by Carathéodory in his Göttingen Dissertation, "*Ueber die diskontinuirlichen Lösungen in der Variationsrechnung.*" The fact that we have three curves meeting in a point makes a separate discussion necessary.

C. Uniqueness of Extremal-Systems Within a Field.

We shall study now more carefully the region in which a discontinuous solution exists.

1. The corner curve, K. We first consider the curve K, which is the locus of the point P_2. From (29),

$$(31)\qquad K: \quad \begin{aligned} y_2 &= \frac{2}{\sqrt{3}}\, a, \\ x_2 &= a_0 + a\left(-\log\sqrt{3} + \operatorname{ch}^{-1}\frac{b_0}{a}\right), \end{aligned}$$

where $\operatorname{ch}^{-1}(b_0/a)$ is positive, and $t_0 = -\operatorname{ch}^{-1}(b_0/a)$. We may write the single equation for K,

$$(32)\qquad K: \quad x_2 - a_0 = \frac{\sqrt{3}}{2}\, y_2\left[\operatorname{ch}^{-1}\frac{2b_0}{\sqrt{3}\,y_2} - \log\sqrt{3}\right],$$

where the positive sign of the function $\operatorname{ch}^{-1}(2b_0/\sqrt{3}\,y_2)$ is to be taken. Here x_2 appears as a single-valued continuous function of y_2 defined for the range $0 < y_2 \leqq b_0$, and x_2 approaches the value a_0 as y_2 approaches zero. Further, if $y_2 = b_0$, we have $x_2 = a_0$.

We compute from (31) the derivatives

$$\frac{dx_2}{dy_2} = \frac{\dfrac{dx_2}{da}}{\dfrac{dy_2}{da}} = \frac{\sqrt{3}}{2}\left\{\operatorname{ch}^{-1}\frac{b_0}{a} - \log\sqrt{3} - \frac{b_0}{\sqrt{b_0^2 - a^2}}\right\},$$

$$(33)\qquad \frac{d^2x}{dy_2 dy_2} = \frac{\dfrac{dy_2}{da}\cdot\dfrac{d^2x_2}{da^2} - \dfrac{dx_2}{da}\cdot\dfrac{d^2y_2}{da^2}}{\left(\dfrac{dy}{da}\right)^3} = -\frac{3b_0^2}{4a(b_0^2 - a^2)^{3/2}} < 0.$$

Hence $\dfrac{dx_2^*}{dy_2}$ is a decreasing function and the curve K is concave to the line $x = a_0$.

Furthermore $\dfrac{dx_2}{dy_2}$ approaches $+\infty$ as y_2 approaches 0, takes the value -2 for $y_2 = b_0$, and the value zero when $\chi(t_0) = \log\sqrt{3}$, where $b_0 = a\operatorname{ch} t_0$.

This value of t_0 is greater than t_0' where $\chi(t_0') = 2 + \log\sqrt{3}$, since $\chi(t)$ is a decreasing function, continuous except at $t = 0$, and t_0 and t_0' are negative. We find approximately $t_0 = -1.63$ and $t_0' = -3.55$.

From (31) the following table, used for plotting the curve K in figure 6, is obtained:

t_0	a	x_2	y_2
-3.55	.060	.180	.070
-3.00	.098	.240	.113
-2.46	.162	.316	.187
-2.00	.262	.380	.302
-1.50	.416	.394	.480
-1.20	.556	.361	.642
-1.00	.652	.293	.753
-0.75	.774	.155	.894
-0.55	.866	0.000	1.000

As t_0 decreases from $-\log\sqrt{3}$ to $-t_0'$, A_2 describes the curve K from A_0 to the fixed point L for which $x_2 = .180$, $y_2 = .070$.

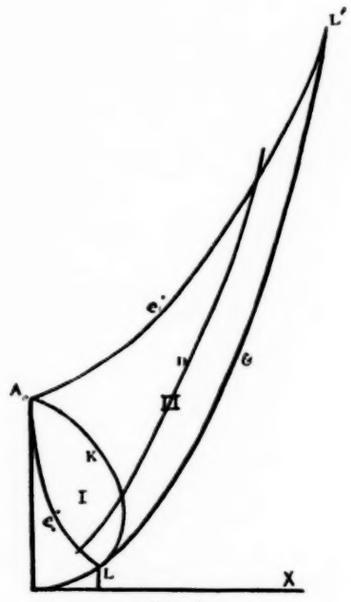

FIG. 6.

2. The envelope E. We now consider the envelope E of the catenaries $\mathfrak{C}_1(a)$, the coordinates of which satisfy the conditions

$$(34) \qquad E: \quad \begin{aligned} x &= a(\tau - t_0 - \log 3) + x_0, \\ y &= a \operatorname{ch} \tau, \end{aligned}$$

where

$$b_0 = a \operatorname{ch} t_0, \qquad \chi(\tau) + \log 3 - \chi(t_0) = 0.$$

Since $\chi(t)$ is a single-valued decreasing function, continuous except at $t = 0$, the equation $\chi(t) = c$ has for any assigned c one and only one negative solution and one and only one positive solution. Hence, in the realm

$$t_0' < t \leqq -\log \sqrt{3}, \qquad \log \sqrt{3} \leqq \tau < t_0',$$

the values x, y, t_0, and τ above are single-valued continuous functions of a. We find

$$\frac{dx}{da} = \coth \tau - \coth t_0 + a\left(\frac{\partial \tau}{\partial a} - \frac{\partial t_0}{\partial a}\right),$$

$$\frac{dy}{da} = \operatorname{ch} \tau + a \operatorname{sh} \tau \frac{\partial \tau}{\partial a}.$$

But

$$\frac{\partial t_0}{\partial a} = -\frac{1}{a} \coth t_0,$$

$$\frac{\partial t}{\partial a} = \frac{\coth^2 t_0 \, \partial t_0}{\coth^2 \tau \, \partial a} = -\frac{\coth^3 t_0}{a \coth^2 \tau}.$$

Hence

$$\frac{dx}{da} = \frac{\coth^3 \tau - \coth^3 t_0}{\coth^2 \tau} > 0,$$

$$\frac{dy}{da} = \frac{\operatorname{sh}^3 \tau}{\operatorname{ch}^2 \tau} (\coth^3 \tau - \coth^3 t_0) > 0,$$

and, as was found above,

$$\frac{dy}{dx} = \operatorname{sh} \tau.$$

The slope of E is therefore everywhere positive. Furthermore

$$\frac{d^2 y}{dx^2} = \operatorname{ch} \tau \frac{d\tau}{dx}.$$

Since

$$\tau = \frac{x - x_0}{a} + \log 3 + t_0,$$

we obtain

$$\begin{aligned}\frac{d\tau}{dx} &= \frac{1}{a} - \frac{x - x_0}{a^2}\frac{da}{dx} + \frac{dt_0}{dx} \\ &= \frac{1}{a}\left[1 - \frac{da}{dx}(\coth t_0 + \coth \tau - \coth t_0)\right] \\ &= \frac{1}{a}\left[1 - \frac{\coth^3 \tau}{\coth^3 \tau - \coth^3 t_0}\right] \\ &= -\frac{1}{a}\frac{\coth^3 t_0}{\coth^3 \tau - \coth^3 t_0}.\end{aligned}$$

Therefore

$$\frac{d^2y}{dx_2} = -\frac{\operatorname{ch}\tau}{a}\frac{\coth^3 t_0}{\coth^3 \tau - \coth^3 t_0} > 0.$$

Hence E is everywhere convex to the x-axis. For the initial values, $a_0 = 0$, $b_0 = 1$, we have $a = 1/\operatorname{ch} t_0$. Using $\chi(t_0), \chi(\tau)$, we compute the following table for points of E:

t_0	$\chi(t_0)$	$\chi(\tau)$	(τ)	a	x	y
− 3.55	2.55	1.45	0.55	.060	0.18	0.07
− 3.00	2.00	0.90	0.71	.098	0.24	0.11
− 2.50	1.49	0.39	0.95	.162	0.42	0.24
− 2.00	0.96	− 0.14	1.29	.262	0.56	0.50
− 1.50	0.40	− 0.65	1.73	.416	0.89	1.20
− 1.20	0.00	− 1.10	2.15	.556	1.25	2.53
− 1.00	− 0.31	− 1.41	2.41	.652	1.53	3.68
− 0.75	− 0.82	− 1.92	2.92	.774	2.00	7.70
− 0.55	− 1.45	− 2.55	3.55	.866	2.60	15.05

The curve E appears in figure 6. It touches the corner curve K in the point $L(x = .180,\ y = .07)$, and the particular curve $\mathfrak{C}_1'$, for which $t_0 = -\log\sqrt{3}$ in the point L' $(x = 2.60,\ y = 15.05)$ where $\tau = \tau_1'$.

3. The field of broken-extremals $[\mathfrak{E}_0(a), \mathfrak{E}_1(a)]$ for which $t_0' < t_0 \leqq -\log\sqrt{3}$. We have now a region I (in figure 6) bounded by the curves $\mathfrak{E}_0'(t_0 = t_0')$ and K, and a second region II (in figure 6) bounded by K, $\mathfrak{E}_1'$, and E. Every line parallel to the x-axis cuts the boundary of I twice (or not at all), once on $\mathfrak{E}_0'$ and once on K, and cuts the boundary of II twice (or not at all), once on K or $\mathfrak{E}_1'$ and once on E, for each of these curves is expressible in the form $x = \phi(y)$, where ϕ is a single-valued function.

The set of extremals $\mathfrak{E}_0(a)$, $(a_0' < a < a_2)$, of the region I, where $a_0' = \dfrac{b_0}{\operatorname{ch} t_0'}$ and $a_2 = \frac{1}{2}\sqrt{3}\, b_0$, are expressible in the form

$$x = x(y, a),$$

where x, x_y, x_a, x_{yy}, x_{ya} are single-valued continuous functions in the region I. Further

$$x_a = \chi(t_0) - \chi(t) > 0 \text{ in I (unless } t = t_0).$$

Consider any point $[x(y, a), y]$ in I. If we give to y any fixed value y_3 of I and let a increase continuously from a' of $\mathfrak{E}_0'$ to $a_2 = \frac{1}{2}\sqrt{3}\, y_3$ at K, $x(y_3, a)$ increases continuously from $x(y_3, a')$ to $x(y_3, a_2)$, and therefore passes once and but once through every intermediate value. Hence, if a_3 be any value of a in (a', a_2) and we put $x(y_3, a_3) = x_3$, then the equation $x(y_3, a) = x_3$ has in (a', a_2) no other solution but $a = a_3$. Through every point (x_3, y_3) of I there passes, therefore one and but one extremal of the set. The region I, is thus a field (improper) of extremals $\mathfrak{E}_0(a)$.*

The set of extremals $\mathfrak{E}_1(a)$, $(a_0' < a < a_2)$, (29), of II is also expressible in the form

$$x = x(y, a),$$

where x, xy, x_a, x_{yy}, x_{ya} are single-valued continuous functions in the region II. Further [see (20)],

$$x_a = \chi(t_0) - \chi(\tau) - \log 3 < 0, \text{ in II}.$$

By an argument similar to the one above we may show that the region II is simply covered by the extremals $\mathfrak{E}_1$.

Under these circumstances, we say that *the region R, composed of the*

* Bolza, loc. cit., §19 *a*, *b*.

regions I *and* II, *constitutes a field simply covered by the broken-extremals* $\mathfrak{E}_0(a)$, $\mathfrak{E}_1(a)$.*

4. Comparison of the system $(\mathfrak{E}_0, \mathfrak{E}_1, \mathfrak{E}_2)$ **with the Goldschmidt solution.**† For the value of the integral over the system $(\mathfrak{E}_0, \mathfrak{E}_1, \mathfrak{E}_2)$ we find

$$J(\mathfrak{E}_0, \mathfrak{E}_1, \mathfrak{E}_2) = 2\pi a^2 \int_{t_0}^{-\log\sqrt{3}} \operatorname{ch} t\, dt + 2\pi a^2 \int_{\log\sqrt{3}}^{\tau_1} \operatorname{ch}^2 \tau\, dt + \tfrac{4}{3}\pi a^2,$$

$$= \frac{\pi}{2} a^2 \left\{ (\operatorname{sh} 2t + 2t) \Big|_{t_0}^{\tau_1} - 4 \log\sqrt{3} \right\}.$$

For the Goldschmidt solution, consisting of the two ordinates through A_0 and A_1 together with the x-axis, the corresponding value is

$$J_G = \pi(b_0^2 + b_1^2) = 4a^2 (\operatorname{ch}^2 t_0 + \operatorname{ch}^2 \tau_1),$$

and consequently

$$(35) \quad J_G - J(\mathfrak{E}_0, \mathfrak{E}_1, \mathfrak{E}_2)$$

$$= \frac{\pi}{2} a^2 \{ 2\operatorname{ch}^2 t_0 + \operatorname{sh} 2t_0 + 2t_0 + 2\operatorname{ch}^2 \tau_1 - \operatorname{sh} 2\tau_1 - 2\tau_1 + 2\log 3 \}$$

$$= \frac{\pi}{2} a^2 \{ 2t_0 + 1 + e^{2t_0} + 2\log 3 - (2\tau_1 - 1 - e^{-2\tau_1}) \}.$$

Let now A_0 be fixed, and A_1 be movable, on a given extremal-system. Then t_0 is fixed, and τ_1 variable, and we consider the function

$$\phi(\tau_1) = 2\tau_1 - 1 - e^{-2\tau_1}.$$

This function has a positive derivative,

$$\phi'(\tau_1) = 2 + 2e^{-2\tau_1} > 0.$$

Hence, as τ_1 increases from its smallest value $\log\sqrt{3}$ to ∞, the difference (35) decreases continuously. On the arc $A_0 L'$ of the curve $\mathfrak{E}_1^*$ we find that if A_1 is coincident with A_0, τ_1 will be equal to $\log\sqrt{3}$ and

$$J_G - J(\mathfrak{E}_0, \mathfrak{E}_1, \mathfrak{E}_2) = \tfrac{2}{3}\pi a^2 > 0.$$

* Compare Carathéodory, loc. cit., §11, where a similar field (fig. 14) appears for a discontinuous solution of the usual kind.

† Goldschmidt, *Prize Essay*, 1830.

If A_1 is coincident with L', τ_1 will be equal to $\tau_1' = 3.55$ and we have

$$J_G - J(\mathfrak{E}_2, \mathfrak{E}_1, \mathfrak{E}_2) < 0,$$

the difference being approximately equal to $-\frac{1}{2}(3.67)(\pi a^2)$. Hence the expression (35) vanishes at some point of $\mathfrak{E}_1'$ between A_0 and L'. The totality of points for which this is true on the extremals $\mathfrak{E}_1$ of the field form a continuous curve D determined by the equations

$$D: \quad \begin{aligned} x &= a\tau_1 - a \log 3 + a_0 - at_0, \\ y &= a \operatorname{ch} \tau_1, \end{aligned}$$

where

$$b_0 = a \operatorname{ch} t_0,$$

and

$$2t_0 + 1 + e^{2t_0} + 2 \log 3 = (2\tau_1 - 1 - e^{-2\tau_1}) = z.$$

For the values, $a_0 = 0$, $b_0 = 1$, we construct the following table for the curve D:

t_0	z	τ_1	$\tau_1 - t_0 - \log 3$	a	x	y
−3.0	−2.80	− .16	1.10	.098	.108	.10
−2.5	−1.79	.08	1.32	.162	.216	.16
−2.0	− .78	.35	1.25	.262	.327	.28
−1.5	.25	.74	1.14	.416	.476	.54
−1.2	.89	1.01	1.11	.556	.618	.87
−1.0	1.34	1.20	1.11	.652	.724	1.18
− .75	1.93	1.49	1.14	.774	.884	1.79
− .50	2.43	1.73	1.18	.866	1.022	2.52

The curve D (see figure 6) divides R into two regions, one adjacent to A_0, in which

$$J_G - J(\mathfrak{E}_0, \mathfrak{E}_1, \mathfrak{E}_2)$$

is positive, and the other adjacent to L', in which this difference is negative. On D itself, J_G and J $(\mathfrak{E}_0, \mathfrak{E}_1, \mathfrak{E}_2)$ are equal.

5. Tallqvist's results. Experimental determination of the limit of stability. In the first of the two papers referred to on page 56, Tallqvist starts from the problem to construct a minimal surface passing

through a given circle and meeting a given plane parallel to the circle at a given angle α. He derives the solution from certain formulae given by H. A. Schwarz.* He then infers from physical considerations that the angle α must be 60°, and finally gives the following rule for the determination of the limit of stability:

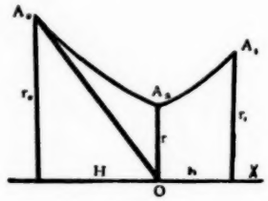

FIG. 7.

Let r_0 and r_1 ($r_0 > r_1$) be the radii of the rings. Let A_0A_2 and A_2A_1, be two catenaries which meet a normal OA_2 to their common directrix, at an angle of 60°. Let H be the projection on the directrix of the film A_0A_2, and h that of A_2A_1. Then H is determined by the condition that the tangent to A_0A_2 at $A_0(y_0 = r_0)$ must pass through the foot of the normal OA_2. This determines the catenaries (see p. 15 above), and h is determined by the point A_1 of the catenary A_2A_1 whose ordinate is r_1.

This leads to the following formulas:

$$\log \frac{2r_0 + \sqrt{4r_0^2 - 3r^2}}{3r} = \frac{2r_0}{\sqrt{4r_0^2 - 3r^2}} = \frac{2H}{r\sqrt{3}},$$

$$h = r\,\frac{\sqrt{3}}{2}\,ln\,\frac{2r_1 + \sqrt{4r_1^2 - 3r^2}}{3r},$$

from which it is found that

$$\frac{r}{r_0} = 0.436047,$$

$$\frac{H}{r_0} = 0.407824.$$

The quotient h/r_0 can be found for any values r_0, r_1 ($r_1 \leqq r_0$).

In the special case where the two rings are equal, Tallqvist's rule agrees exactly with our own result given in A, §7. In the general case, however, for the limit of stability when the two rings are unequal, Tallqvist's rule gives

* H. A. Schwarz, *Miscellen aus dem gebiete der Minimalflächen.*

a different value from the results obtained in A, §9.* Tallqvist's experiments for this case are not decisive as the two rings which he used were too nearly equal in size. We give below his results for unequal rings compared with his formulas and the results found from the formulae of the present paper, where b_0 and y are the radii of the rings:

	b_0	y	$x - a_0 = H + h$
Tallqvist's experimental values (21 trials)	22.455 *mm*	22.402 *mm*	18.432
	1.	.99766	.82086
Tallqvist's theoretical values	22.455	22.402	18.296
	1.	.99766	.81480
Theoretical values from tables given above	1.	.99766	.81480

Since these results are inconclusive it seemed desirable to carry out a new series of experiments in which the two rings should be of quite different diameter. The experiments were conducted as follows (see figure 8):

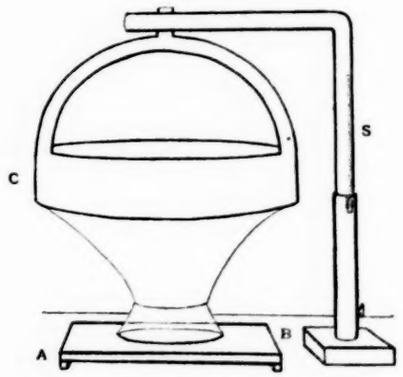

FIG. 8.

A square plate of brass AB with a circular opening, the latter beveled to a knife-edge at the upper surface, was put in horizontal position by means of a spirit level and held rigidly in place. Above this was suspended a ring C

* Compare also C §2, formulas (34).

made from a band of heavy brass, beveled at the bottom on the inside to a knife-edge. This ring was attached to the arm of a stand S, which was provided with a screw to give smooth vertical movement and a vernier scale to ensure accurate measurement.

The upper face of the brass plate was covered with a film, the ring lowered to it and withdrawn. At first there appeared beside the horizontal film across the circular opening of the plate a single catenoid extending from the ring to the plate. When the ring was withdrawn so far that the lower margin of the catenoid became as small as the circular opening, the films were transformed into the desired system. The height was then adjusted until the limit of stability was found. The diameters of the two circles were measured with calipers. The following are the results, where D_0 and D_1 are the diameters of the two rings, H the distance between them, or the height of the film:

	D_0	D_1	H_1
	87.7 *mm*	51.15 *mm*	27.3 *mm*
	87.8	51.15	27.25
	87.4	51.10	27.25
	87.0	51.20	27.3
	87.1	51.10	27.4
	87.6	51.15	27.35
	87.8	51.15	27.4
	87.7	51.15	27.35
	87.0	51.1	27.35
	87.5	51.15	27.3
			27.5
			27.3
			27.35
			27.3
			27.3
			27.3
Average	87.52	51.14	27.33

Then, if R_0 and R_1 represent the radii of the two circles, and if we reduce to the scale for which $R_0 = 1$, we have $R_0 = 1$, $R_1 = .5844$, and $H_1 = .6244$.

This value of H_1 can readily be compared with that given by formulas C, §2 (34). We have

$$a_0 = 0, \qquad b_0 = 1 = a \operatorname{ch} t_0,$$

$$\chi(t_0') - \chi(t_0) + \log 3 = 0.$$

Then

$$y = .5844 = a \operatorname{ch}(t_0'),$$

$$x = H_1 = a(t_0' - t_0 - \log 3).$$

These equations have, for t_0 negative and t_0' positive, a unique solution, found by the method of approximation:

$$a = .2835, \quad t_0 = 1.933, \quad t_0' = 1.3515, \quad x = H_1 = .6195.$$

Tallqvist's formulas give $x - a_0 = H + h = .5795$. By the formulas here given, the error is 0.8 %, by Tallqvist's 7.2 %. From these results, it may be inferred that Tallqvist's theoretical determination of the limit of stability is wrong, and that the minimum problem which nature solves in the experiment of the two rings is actually the one which we have formulated in the introduction.

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ON THE GEOMETRIC PROPERTIES OF QUARTIC CURVES POSSESSING FOURFOLD SYMMETRY WITH RESPECT TO A POINT*

By R. D. Carmichael

In a previous paper† I have considered the problem of classifying, by the form of their equations, plane algebraic curves possessing fourfold symmetry with respect to a point. The center of symmetry being taken as the origin, it is there shown that all such curves are of two classes: and for quartic curves it readily follows that the equations of the two classes in their most general forms are respectively:—

$$(A)\quad 2a_0(x^4+y^4)+4a_1xy(x^2-y^2)+4a_2x^2y^2+a_3(x^2+y^2)+a_4=0,$$

$$(B)\quad r_0(x^4-y^4)+2r_1(x^2+y^2)+r_2(x^2-y^2)+2r_3xy=0.$$

In this paper attention will first be given to the classification of the irreducible forms of these curves according to the number of their singularities of each kind. For this purpose we shall require certain of Plücker's equations. By n, m, δ, τ, ρ, ι, we shall, as usual represent respectively the order, class, number of double points, number of double tangents, number of cusps, and number of points of inflection of the curve. The Plücker equations are then the following:—

$$(1)\qquad m=n(n-1)-(2\delta+3\rho),$$

$$(2)\qquad n=m(m-1)-(2\tau+3\iota),$$

$$(3)\qquad \iota=3n(n-2)-(6\delta+8\rho),$$

$$(4)\qquad \rho=3m(m-2)-(6\tau+8\iota).$$

From the fact that these curves possess fourfold symmetry with respect to the origin it follows that singularities not at the origin can enter only by fours.

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* Read before the American Mathematical Society, October 26, 1907.

† Annals of Mathematics, ser. 2, vol. 9, pp. 53-56; January 1908.

Moreover, even at the origin if one cusp exists there must be three others corresponding to it; for, if not, the curve would not coincide with its former position after one-fourth revolution about the origin. Hence the number of cusps is a multiple of four, or zero. But since $m + 2\delta$ is positive and n is four it follows from (1) that ρ is less than four. Hence ρ is equal to zero; that is, the curves in consideration are non-cuspidal. Now, a quartic curve has not more than three double points; and therefore these in consideration can have no double point except at the origin.

In group A, if $a_4 \neq 0$, the locus does not pass through the origin, and hence the curve has no double point. If $a_4 = 0$ and $a_3 \neq 0$, there is a double point at the origin. If both $a_4 = 0$ and $a_3 = 0$, the equation becomes homogeneous in x, y and hence breaks up into linear factors. The locus is then a group of straight lines and is therefore excluded from further consideration. In group B there is always a double point at the origin if r_2 and r_3 are not both zero: and if $r_2 = r_3 = 0$, the equation breaks up into factors and the curve is reducible. If there is no double point, that is, if $\delta = 0$, it follows from equations (1) to (4) that $m = 12$, $\iota = 24$, $\tau = 28$. If there is one double point, or $\delta = 1$, we have $m = 10$, $\iota = 18$, $\tau = 16$. Tabulating we have:

	n	m	δ	ρ	τ	ι
Group A: Case I, $a_4 \neq 0$	4	12	0	0	28	24
Group A: Case II, $a_4 = 0$	4	10	1	0	16	18
Group B	4	10	1	0	16	18

Since δ is in no case greater than one, none of these curves is unicursal. The first one belongs to Salmon's group I and the other two to his group II.*

We shall take up these cases separately and determine for each the general form of the locus and certain other geometric properties in addition to those already given.

Group A, Case I. Making the substitution $x = \rho \cos\theta$ and $y = \rho \sin\theta$, and reducing to more convenient form, we have the following polar equation:

$$(5) \qquad [2a_0 + (a_2 - a_0)\sin^2 2\theta + a_1 \sin 4\theta]\,\rho^4 + a_3\,\rho^2 + a_4 = 0.$$

* See Salmon-Fiedler's *Analytische Geometrie der höheren ebenen Kurven*, second ed., p. 278.

The coefficient of ρ^4 may be written in the form

$$2a_0 + \frac{a_2 - a_0}{2} + a_1 \sin 4\theta + \frac{a_0 - a_2}{2} \cos 4\theta,$$

which by the substitution of $\theta - \phi$ for θ becomes

$$2a_0 + \frac{a_2 - a_0}{2} + \sin 4\theta\left(a_1 \cos 4\phi + \frac{a_0 - a_2}{2} \sin 4\phi\right)$$
$$+ \cos 4\theta\left(\frac{a_0 - a_2}{2} \cos 4\phi - a_1 \sin 4\phi\right).$$

Putting

$$\tan 4\phi = \frac{a_0 - a_2}{2a_1}$$

in the above expression and substituting in (5), we may write the result in the form

$$(A_1 + B_1 \sin 4\theta)\ \rho^4 + C_1\rho^2 + a_4 = 0, \tag{6}$$

where $A_1 = 2a_0 + \frac{a_2 - a_0}{2}$ and $B_1 = \frac{4a_1^2 + (a_0 - a_2)^2}{2a_1} \cos 4\phi.$

Since $a_4 \neq 0$ we may divide by it and write the result thus:

$$(A + B \sin 4\theta)\ \rho^4 + C\rho^2 + 1 = 0,$$

where $A = \frac{A_1}{a_4}, \quad B = \frac{B_1}{a_4}, \quad C = \frac{C_1}{a_4}.$

If $C = 0$ the equation reduces to

$$\rho = \pm \sqrt[4]{\frac{-1}{A + B \sin 4\theta}}.$$

The discussion is entirely analogous to that for the general equation in Case II of Group A below. It is unnecessary to give it in detail.

If $C \neq 0$, we have

$$\rho = \pm \sqrt{\frac{-C \pm \sqrt{C^2 - 4r}}{2r}}, \tag{7}$$

where $r = A + B \sin 4\theta$. The two values of ρ corresponding to θ and $\theta + 180°$ are evidently the same, and consequently the first double sign in (7) may be omitted and the resulting equation will give, when θ varies from 0° to 360°, every point of the locus which is obtained when the double sign is kept. Hence we may write

$$\rho = \sqrt{\frac{-C \pm \sqrt{C^2 - 4r}}{2r}}. \tag{8}$$

If $r < 0$, there is one real value of ρ.

If $r = 0 -$ *, there is a finite real value of ρ when $C < 0$, and an infinite real value of ρ when $C > 0$.

If $r > C^2/4$, there is no real value of ρ.

If $r = C^2/4$, there are two coincident values of ρ when $C < 0$, but no real value when $C > 0$.

If $r > 0$ but $< C^2/4$, there are two real and distinct values of ρ when $C < 0$, but no real value when $C > 0$.

If $r = 0 +$, there is a finite real value and an infinite real value of ρ when $C < 0$, but no real value when $C > 0$.

Now, if B is positive,

$$A - B \leqq r \leqq A + B.$$

If B is negative it should be replaced by $-B$ in the following discussion. A discussion of the different possible figures in terms of A, B, C and hence in terms of the a's offers but little difficulty.

1) If $A < -B$, r is always negative and the locus is a simple closed curve.

2) If $A = -B$, r is negative or zero. When $r \neq 0$ there is a single finite real ρ for every θ. When $r = 0 -$ there is a finite value of ρ if $C < 0$, and the locus is a closed curve. If $C > 0$, the curve has infinite branches, of which the asymptotes are given by $r = 0$.

3) If $B > A > -B$, r has both positive and negative as well as zero values. If $C > 0$ we have a single branch in each quadrant, corresponding to a negative r, and approaching as asymptotes the lines $r = 0$. No real points correspond to values of θ for which r is positive.

* That is, approaches zero through negative values.

That the lines $r = 0$ are asymptotes of the curve in this case and in the preceding case may be proved by showing that the distance of a variable point (ρ, θ) of the curve from the line $r = 0$, which is $\rho \sin(\alpha - \theta)$, where α is the value of θ for which $r = 0$, approaches zero as θ approaches α. We have

$$A + B \sin 4\theta = r, \qquad A + B \sin 4\alpha = 0,$$
$$r = B(\sin 4\theta - \sin 4\alpha) = 2B \sin 2(\theta - \alpha) \cos 2(\theta + \alpha).$$

Then $\rho \sin(\alpha - \theta)$ approaches zero as $[\sin(\alpha - \theta)]^{\frac{1}{2}}$, unless $\cos 2\alpha = 0$ or $B = 0$. If $B = 0$ we have also $A = 0$ and the curve is excluded from our consideration, being of the second degree; if $\cos 2\alpha = 0$, we have $A = 0$ and $\sin 4\alpha = 0$, so that $r = B \sin 4\theta$. It may be seen that this case is no exception, but that here also $\rho \cos(\alpha - \theta)$ approaches zero as $[\sin(\alpha - \theta)]^{\frac{1}{2}}$.

To discuss the form of the curve when C is negative, consider first a value of θ for which r is negative, and let θ change through 90° so that r increases to zero, becomes positive, then decreases to zero, becomes negative, and returns to its original value. There is at first one value of ρ remaining finite as $r = 0-$; when r increases through zero there is a finite value and an infinite value $(r = 0+)$, so that the finite branch is continued and a second branch has the line $r = 0$ as an asymptote. That $r = 0$ gives an asymptote may be proved as before. If r remains less than $C^2/4$, these branches remain distinct, the infinite branch having a second asymptote when $r = 0+$, so that the locus is a closed curve and four infinite branches like those of a hyperbola. If, however, r becomes greater than $C^2/4$, the two branches come together and have a common tangent from the origin for $r = C^2/4$, and for values greater than $C^2/4$ no real values of ρ. If $A + B = C^2/4$, so that $C^2/4$ is the maximum value of r, we should apparently have a double point, not at the origin, and so contradictory to our previous result. It may, however, be easily seen that in this case the locus is not irreducible, but consists of two hyperbolas. It is of course not proved that the curve is irreducible in other cases.

4) If $A = B$, r is equal to or greater than zero. If $C > 0$, there is no real locus. If $C < 0$, there are two distinct values of ρ, one becoming infinite with $r = 0$, and the locus is as in the previous case where $A + B < C^2/4$.

5) Finally, if $A > B$, $r > 0$. When $C > 0$, the curve has no real locus. When $C < 0$, we have two finite values of ρ, always distinct unless $A - B \leqq C^2/4 \leqq A + B$. If $C^2/4$ lies outside of these limits the locus consists of two separate closed curves, each surrounding the origin; if between

$A - B$ and $A + B$, the curve consists of four separate closed curves, each excluding the origin, whose tangents from the origin are given by $r = C^2/4$; if finally $C^2/4$ is equal to $A - B$ or $A + B$, the locus reduces to two hyperbolas.

Group A, Case II. Putting $a_1 = 0$ in equation (6), and dividing by $-C_1$, we may write as the polar equation of the curves to be considered here,

$$(A + B \sin 4\theta)\, \rho^4 - \rho^2 = 0, \tag{9}$$

where the values of A and B are easily found in terms of the a's. From (9) follows

$$\rho^2 = 0, \tag{10}$$

$$\rho = \sqrt{\frac{1}{A + B \sin 4\theta}}. \tag{11}$$

Corresponding to the values of ρ given by (10) is simply the origin, an isolated point. In (11) the double sign has been omitted for the same reason as that given in the discussion leading to (8).

We have seen that this locus has 18 points of inflection. Moreover, those not at the origin enter by fours. Hence the isolated point at the origin must also count as at least two points of inflection. If there are more than two points of inflection at the origin there must be six, so as to leave those not at the origin a multiple of four. But six coincident points on a quartic is impossible; and therefore the origin counts as just two points of inflection.

Again setting $r = A + B \sin 4\theta$, it will not be difficult to discuss the locus of (11) in terms of A and B. We have as before

$$A - B \leqq r \leqq A + B,$$

if B is positive, with the same modification as before if B is negative.

1) If $A \leqq -B$, there is no real locus.

2) If $B > A > -B$, r has both positive and negative values. Negative values give no real points. Positive values of r give the locus consisting of four branches, each like that of a hyperbola and having two asymptotes given by $r = 0$.

3) If $A = B$, the locus consists, as in the previous case, of four branches

extending to infinity, each having an asymptote common with the next, so that the asymptotes are along two perpendicular lines.

4) Finally, if $A > B$, we have $r > 0$, and there is then a single closed branch.

Group B. Transforming the equation by the substitution $x = \rho \cos \theta$ and $y = \rho \sin \theta$, and reducing, we have:

$$(r_0 \cos 2\theta + r_1 \sin 2\theta)\ \rho^4 + (r_2 \cos 2\theta + r_3 \sin 2\theta)\ \rho^2 = 0.$$

Replacing θ by $\theta - \phi$ and setting

$$\tan 2\phi = \frac{r_0}{r_1},$$

the equation takes the form

$$\rho^4 A_1 \sin 2\theta + \rho^2(B_1 \sin 2\theta + C_1 \cos 2\theta) = 0,$$

where

$$A_1 = r_0 \sin 2\phi + r_1 \cos 2\phi,$$
$$B_1 = r_2 \sin 2\phi + r_3 \cos 2\phi,$$
$$C_1 = r_2 \cos 2\phi - r_3 \sin 2\phi,$$

If $A_1 = 0$ this equation is of the second degree and need not be considered. If $A_1 \neq 0$ we may replace the equation by

$$\rho^4 \sin 2\theta + \rho^2(B \sin 2\theta + C \cos 2\theta) = 0,$$

where $$B = B_1/A_1 \text{ and } C = C_1/A_1.$$

This equation is equivalent to the two

$$\rho^2 = 0,$$

$$\rho = \pm \sqrt{-B - C \operatorname{ctn} 2\theta}.$$

The first of these equations gives only the origin. The second is a circle, real or imaginary, if C is zero. If C is not zero the locus consists of two real branches, each having an inflexion at the origin and approaching both ends of one of the coordinate axes asymptotically.

ANNISTON, ALA.,
OCTOBER, 1907.

EXTENSION OF THE SIEVE OF ERATOSTHENES TO ARITHMETICAL PROGRESSIONS AND APPLICATIONS*

By J. C. Morehead

Almost without exception the methods employed in the elaboration of the various existing factor-tables are modifications, more or less remote, of the classic Sieve of Eratosthenes. In most instances these factor-tables are designed to show the prime numbers and the prime factors of the composite numbers occurring in sets of consecutive integers.† The Sieve method, as applied to the determination of the primes in a set of consecutive integers

$$m, \quad m+1, \quad m+2, \quad m+3, \cdots m+n,$$

may be briefly described as follows: We strike out from the numbers of the set the multiples of 2: from the remaining numbers of the set the multiples of 3; then, from those yet remaining, the multiples of 5, and so continue, for the multiples of all primes not greater than $\sqrt{(m+n)}$. The numbers finally remaining in the set are all prime. If, as the procedure just described is carried out, the multiples of 2, 3, 5, $\cdots$ are conveniently indicated, we obtain what is called a *factor-table* for the set of numbers considered.

The present paper describes an analogous method applied to successive numbers in arithmetical progression. The discussion in §§1–5 is quite general, while the remaining part of the paper is occupied with the discussion of applications and special cases.

1. Extension of the Sieve method. Any arithmetical progression of m integral terms may be written, in direct or inverse order, in either of the forms

$$(1) \qquad a^k + b, \quad 2a^k + b, \quad 3a^k + b, \cdots \quad ma^k + b,$$

$$(1') \qquad -a^k + b, \quad -2a^k + b, \quad -3a^k + b, \cdots -ma^k + b,$$

* Read before the American Mathematical Society, April 29, 1905; December 29, 1905.

† The Factor-Tables of J. Burckhardt, Z. Dase and J. Glaisher applying to all integers under 9000000, their recent extension by L. D. Lehmer, under the auspices of the Carnegie Institution, Washington, D. C., to 10000000, and the tables of "High Primes" of A. Cunningham and H. J. Woodall are the principal tables of the sort that have been satisfactorily verified.

if suitable integral values are assigned to the symbols, a, b, k, m. The following general discussion may be limited, therefore, to progressions of the forms (1) and (1′). Evidently we may assume, without loss of generality, that a is positive and k positive or zero. We shall further assume, unless the contrary is expressly stated, that b is prime to a^k;* for each term in (1) and (1′) would admit the common divisor of b and a^k as a factor, — a case which it is useless to include in a search for prime numbers or in the formation of a factor-table.

The methods and results obtained in this paper have for basis and starting point the following

THEOREM. *Let p be any prime number that does not divide a^k, and let the least non-negative solution of the congruence*

$$xa^k + b \equiv 0 \bmod p,$$

for assigned values of a, b and k, be

$$x = x_{kp}.$$

Then only the

$$(x_{kp})^{\text{th}}, \quad (p + x_{kp})^{\text{th}}, \quad (2p + x_{kp})^{\text{th}}, \cdots (pm_p + x_{kp})^{\text{th}}$$

numbers in (1) *are divisible by p and only the*

$$(p - x_{kp})^{\text{th}}, \quad (2p - x_{kp})^{\text{th}}, \quad (3p - x_{kp})^{\text{th}}, \cdots (pm'_p - x_{kp})^{\text{th}}$$

numbers in (1′) *are divisible by p, m_p and m'_p being the maximum integers such that*

$$pm_p + x_{kp}, \quad pm'_p - x_{kp} \leqq m.\dagger$$

As here defined, x_{kp} is clearly a function of k, p, a and b. This will occasionally be indicated by writing x_{kpab}. However, when it is believed that no ambiguity or confusion will arise the symbol will always be abbreviated to x_{kp} or even x_k.

The above theorem leads at once to the following

---

* An exception is made in the application of the relation (M) in §11.

† From well known properties of linear congruences it follows that no solution x_{kp} exists when p is a factor of a^k; but for all other prime values of p an unique solution exists such that $0 \leqq x_{kp} \leqq p - 1$, and that all other solutions are represented by

$$x = np + x_{kp}, \; n = \pm 1, \pm 2, \pm 3, \cdots.$$

The validity of the theorem is thus immediately rendered apparent.

EXTENSION OF THE SIEVE OF ERATOSTHENES. *Let π be the greatest prime not greater than $\sqrt{M}$, where M is the maximum numerical value of the numbers in* (1) *and* (1′) *for an assigned set of values of a, b, k and m. Let the values of x_{kp} be known for all prime values of p as far as π. Then to determine the primes in* (1) *we strike out from* (1) *the*

$$(x_{k2})^{\text{th}},\quad (2+x_{k2})^{\text{th}},\quad (2\times 2+x_{k2})^{\text{th}},\quad (3\times 2+x_{k2})^{\text{th}},\ \cdots\ (2m_2+x_{k2})^{\text{th}}$$

numbers, which are multiples of 2; then the

$$(x_{k3})^{\text{th}},\quad (3+x_{k3})^{\text{th}},\quad (2\times 3+x_{k3})^{\text{th}},\quad (3\times 3+x_{k3})^{\text{th}},\ \cdots\ (3m_3+x_{k3})^{\text{th}}$$

numbers, which are multiples of 3; then the

$$(x_{k5})^{\text{th}},\quad (5+x_{k5})^{\text{th}},\quad (2\times 5+x_{k5})^{\text{th}},\quad (3\times 5+x_{k5})^{\text{th}},\ \cdots\ (5m_5+x_{k5})^{\text{th}}$$

numbers, which are divisible by 5; and so continue, until finally we strike out from the remaining numbers the

$$(x_{k\pi})^{\text{th}},\quad (\pi+x_{k\pi})^{\text{th}},\quad (2\pi+x_{k\pi})^{\text{th}},\quad (3\pi+x_{k\pi})^{\text{th}},\ \cdots\ (\pi m_\pi+x_{k\pi})^{\text{th}}$$

numbers, which are multiples of π; the numbers now remaining are the primes occurring in (1).

Similarly, to determine the primes in (1′), *we strike out from* (1′) *the*

$$(2-x_{k2})^{\text{th}},\quad (2\times 2-x_{k2})^{\text{th}},\quad (3\times 2-x_{k2})^{\text{th}},\ \cdots\ (2m_2-x_{k2})^{\text{th}}$$
$$(3-x_{k3})^{\text{th}},\quad (2\times 3-x_{k3})^{\text{th}},\quad (3\times 3-x_{k3})^{\text{th}},\ \cdots\ (3m_3-x_{k3})^{\text{th}}$$
$$\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots\quad\cdots$$
$$(\pi-x_{k\pi})^{\text{th}},\quad (2\pi-x_{k\pi})^{\text{th}},\quad (3\pi-x_{k\pi})^{\text{th}}\ \cdots\ (\pi m_\pi-x_{k\pi})^{\text{th}}$$

numbers, and those finally remaining in (1′) *are prime.*

If, as this process is carried out, the multiples of 2, 3, 5, 7, $\cdots\ \pi$ *are indicated, we obtain a factor-table for the progressions* (1) *and* (1′).

When $x_{k2}, x_{k3}, x_{k5}, \cdots x_{k\pi}$ are known, this "sifting" is essentially identical with the ordinary Sieve. In order, therefore, to render this extension of the Sieve method in any degree practicable, it is necessary to devise means of calculating expeditiously the numbers x_{kp} for successive primes and for assigned values of a, b and k.

2. The calculation of x_{kp}. Connection with residue tables. The choice of the most convenient means of computing x_{kp}, *i. e.*, of solving

the congruence

$$(A) \qquad xa^k + b \equiv 0 \bmod p,$$

depends, in general, on the relative values of k, a and p. The direct solution of (A), even for small p and for the few cases where residue tables are available giving the least remainders (mod p) of a^k, is quite tedious. However, when such tables are at hand, we express the solution of (A) in the form

$$(B) \qquad x_{kp} \equiv -\frac{b}{a^k} \equiv -ba^{\nu-k} \equiv -ba^{\nu-k'} \bmod p,$$

where ν is the principal exponent* of a (mod p), and k' is the least positive remainder of k (mod ν), take the least remainder, R, of $a^{\nu-k'}$ from the tables, and compute directly x_{kp} as the least positive remainder (mod p) of $-bR$. When $b = \pm 1$, we may thus read the successive values of x_{kp} directly from the residue tables.† Thus for $a = 6$, $b = -1$, $p = 13$, $k = 8$, we have $x_8 \equiv 6^{12-8} \equiv 7 \bmod 13$.

3. Calculation of x_{k+1} from x_k. First method. In many cases, especially when series for different values of k, but fixed values of a and b, are under investigation, it is found expedient to employ one of the following methods of deriving $x_{k+1,p}$ from $x_{k,p}$, $x_{k+2,p}$ from $x_{k+1,p}$, and so on, beginning with a known x_{kp}. Setting $k = 0$ in (B), we obtain

$$(C) \qquad x_{0p} \equiv -b \bmod p.$$

Hence x_{0p} is in every case known or readily determined, and therefore the calculation of x_k for successive values of k may be begun with x_0.

Replacing x by x_k, and writing (A) in the form

$$\frac{x_k}{a} a^{k+1} + b \equiv 0 \bmod p,$$

we have at once

$$(D) \qquad x_{k+1} \equiv \frac{x_k}{a} \equiv \frac{x_k + yp}{a} \bmod p,$$

* The minimum integer, ν, such that $a^\nu \equiv 1 \bmod p$.

† The only printed residue table of sufficient extent to be of use in this connection is *A Binary Canon*, by Lt.-Col. A. Cunningham. This table gives the least positive remainders of 2^x for $x = 1, 2, 3, \cdots \nu$, and for all primes under 1000. An extension of the table (as yet in MS.), by Col. Cunningham and Mr. H. J. Woodall, includes all primes under 10000 for $x \leqq 100$, and all primes under 12000 for $x \leqq 36$. Hence when $a = 2$, $p < 12000$, and k is such that $\nu - k'$ falls within the range of values of x just stated, the *Binary Canon* is directly applicable to the computation of x_{kp}.

where y is an arbitrary integer. Since a and p are mutually prime, an unique choice of y from the numbers $0, 1, 2, \cdots a-1$ is possible, such that $(x_k + yp)/a$ is integral and less than p. When y is so chosen,* we may write instead of (D)

$$(E) \qquad x_{k+1} = \frac{x_k + yp}{a}.$$

For example, when $a = 2$, $y = 0$ or 1 according as x_k is even or odd; and therefore the computation of x_{k+1} from x_k involves simply division by 2 when x_k is even, or the addition of p and then division by 2 when x_k is odd. Thus, for $a = 2$, $b = 1$ and $p = 37$, $x_0 = 36$; $x_1 = 36/2 = 18$; $x_2 = 9$; $x_3 = (9 + 37)/2 = 23$; $x_4 = (23 + 37)/2 = 30$; etc.

Again, when $a = 10$, y may be chosen at sight so that the final digit of $yp + x_k$ will be zero.† Thus, for $a = 10$, $b = 3$, and $p = 43$, we have $x_0 = 40$; $x_1 = 4$; $x_2 = (4 + 2 \times 43)/10 = 9$; $x_3 = (9 + 7 \times 43)/10 = 31$; etc.

4. Calculation of x_{k+1} from x_k. Second method. For brevity, set $\xi_k = x_{kpa1}$ and $\xi'_k = x_{kpa-1}$. The following relations are always satisfied, ξ_k, ξ'_k and x_k referring to the same a and p.

$$\left.\begin{aligned}
(F) \qquad & x_{k+1} \equiv -\xi_1 x_k \\
(F') \qquad & x_{k+1} \equiv \xi'_1 x_k \\
(G) \qquad & x_{k+h} \equiv (-\xi_1)^h x_k \\
(G') \qquad & x_{k+h} \equiv \xi_1'^h x_k \\
(H) \qquad & x_{k+h} \equiv -\xi_h x_k \\
(H') \qquad & x_{k+h} \equiv \xi'_h x_k \\
(J) \qquad & x_{k+nh} \equiv (-\xi_h)^n x_k \\
(J') \qquad & x_{k+nh} \equiv \xi_h'^n x_k
\end{aligned}\right\} \bmod p.$$

* The selection of y involves the solution of the congruence $yp + x_k \equiv 0 \bmod a$, the difficulty of which, compared with the labor of solving (A) directly, will generally depend on the relative values of a, p and k. The application of (E) is essentially the same for k small or large, which is not the case with the solution of (A). Also, if $p > a$, the application of (E) is clearly the preferable method.

† In the case $a = 10$ the use of ordinary commercial *adding machines* greatly facilitates the de ivation of x_{k+1} from x_k; for the process reduces to the addition of p to x_k repeated until the final digit of the sum is zero, and then dropping the cipher. This procedure may be modified so as to be applicable to certain other special cases.

Evidently (F), (G) and (H) are contained in (J), and (F'), (G') and (H') are contained in (J'). All, however, are useful in computation, and are written together for convenient reference.

The demonstration of (J) and (J') establishes the validity of the six preceding relations as well. From the definitions of x_k and ξ_h we have

$$(\gamma) \qquad -\xi_h \equiv a^{-h} \bmod p.$$

Multiplying (c) by (γ) n times, member for member, we obtain

$$(-\xi_h)^n x_k \equiv -ba^{-(k+nh)} \bmod p.$$

But by definition

$$x_{k+nh} \equiv -ba^{-(k+nh)} \bmod p;$$

therefore

$$x_{k+nh} \equiv (-\xi_h)^n x_k \bmod p.$$

The demonstration of (J') is similar.

The computation of x_{k+1}, x_{k+h}, or x_{k+nh} from x_k by means of the above relations involves only the *direct* calculation of the least positive remainder (mod p) of a product. Thus, for $p = 41$, $a = 14$ and $b = 5$, we have, from (C) and (E),

$$\xi_0' = 1, \quad \xi_1' = (1 + 41)/14 = 3; \quad x_0 = 36,$$

and therefore, by (F'),

$$\left.\begin{aligned} x_1 &\equiv 3 \times 36 \equiv 26 \\ x_2 &\equiv 3 \times 26 \equiv 37 \\ x_3 &\equiv 3 \times 37 \equiv 29 \\ x_4 &\equiv 3 \times 29 \equiv 5 \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}\right\} \bmod 41.$$

Thus (F) and (F') are admirably adapted to the compution of x_k for successive values of k, yielding a method generally practicable. For a few special values of a, however, the method of §3 involves less reckoning and is therefore preferable. Any one of the formulas (G), (G'), $\cdots$ (J') may be employed in calculating x_k for non-consecutive values of k, — in practice, that one is selected with which the smallest ξ or ξ' may be used. Applied in this

way at intervals, the congruences (G), (G'), $\cdots$ (J') are especially valuable as check formulas.

5. Relation between x_{kpab} and x_{kpa-b}. For brevity, we write $x_k = x_{kpab}$ and $x'_k = x_{kpa-b}$. By definition

$$x_k a^k + b \equiv 0 \bmod p,$$

$$x'_k a^k - b \equiv 0 \bmod p.$$

By addition, we obtain

$$(x_k + x'_k)\, a^k \equiv 0 \bmod p;$$

hence

$$x_k + x'_k \equiv 0 \bmod p,$$

and since $x_k,\ x'_k < p$, we have at once

$$(K) \qquad x_k + x'_k = p,$$

except when $x_k = x'_k = 0$, *i. e.*, when b is divisible by p. This exceptional case, b divisible by p, requires no computation, and may therefore be excluded from discussion in this and the following section.*

If b is replaced by $-b$, formula (C) becomes

$$(C') \qquad x'_{0p} \equiv b \bmod p,$$

which could have been inferred also from (K). It follows as an immediate consequence of the definition of x'_k, that the same methods and rules that apply to the computation of x_k apply equally well to the computation of x'_k, starting from the initial value given by (C').

When one of the numbers x_k, x'_k is known, the other may be obtained by the relation (K); however, when a table of successive values of x_k and x'_k together is being formed, as illustrated in §6, (K) is employed most effectively as a check formula.

6. Calculation of x_k and x'_k when $a = 2$. As has been noted in §4, the calculation of x_{k+1} from x_k by means of formula (E) is extremely simple when $a = 2$, involving only division by 2 when x_k is even, and the

* The relation (K) is contained implicitly in the theorem of §1, as may easily be seen if we note that the numbers in the series (1) and (1'), with the signs changed throughout, refer to $-b$. In fact, the x_k for the numbers in (1'), with the opposite signs, is simply $p - x_k$, *i. e.*, x'_k.

addition of p and division by 2 when x_k is odd. Even greater simplicity is attained when the computation of successive values of x_k and x'_k is performed simultaneously by a method based on the relations:*

$$(L) \qquad x_k + x'_{k+1} = x_{k+1}, \text{ when } x_k \text{ is odd,}$$

$$(L') \qquad x'_k + x_{k+1} = x'_{k+1}, \text{ when } x'_k \text{ is odd.}$$

We shall demonstrate only (L), as the demonstrations of (L) and (L') are similar in form.

By definition of x_k, we have

$$x_k a^k + b \equiv 0 \bmod p,$$

and multiplying by 2,

$$(k) \qquad x_k a^{k+1} + 2b \equiv 0 \bmod p.$$

Also, by definition of x'_{k+1},

$$(k') \qquad x'_{k+1} a^{k+1} - b \equiv 0 \bmod p.$$

Adding (k) and (k'), we obtain

$$(x_k + x'_{k+1}) a^{k+1} + b \equiv 0 \bmod p,$$

and therefore

$$(l) \qquad x_k + x'_{k+1} \equiv x_{k+1} \bmod p.$$

In keeping with the conventions stated in §1, $p = 2$ is excluded when $a = 2$, and hence it follows from (K) that one of the numbers x_k, x'_k is even and the other odd. Let x_k be odd; then x'_k is even and $x'_{k+1} = (x'_k)/2$. But, from (K), $x_k + x'_k = p$, and hence $x_k + x'_{k+1} = x_k + (x'_k)/2 < p$. We may therefore, when x_k is odd, replace the congruence (l) by the equation

$$x_k + x'_{k+1} = x_{k+1}.$$

The formulas (L), (L') and (E) yield the following compact method for the calculation of successive values of x_k and x'_k when $a = 2$.

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* (L) and (L') are valid only for $a = 2$. The generalizations take the forms

$$(m) \qquad x_k + (a-1)x'_{k+1} \equiv x_{k+1} \bmod p,$$

$$(m') \qquad x'_k + (a-1)x_{k+1} \equiv x'_{k+1} \bmod p,$$

respectively. These congruences are not especially useful in computation when $a > 2$, since when one of the numbers x_{k+1}, x'_{k+1} is known, the other can be found from (K) more readily than from (m) or (m').

The numbers x_0 and x_0' are known and one is even; hence one of the numbers x_1, x_1' can be obtained by a single direct division by 2, and the other by the application of (L) or (L'), involving the addition of two numbers, each of which is less than p: then one of the numbers x_2, x_2' can be obtained by a single direct division by 2, and the other by the application of (L) or (L'); and so on, for x_3, x_3'; x_4, x_4'; · · ·.

The compactness of the method and the speed it is possible to attain in the reckoning are perhaps best understood in connection with a few examples arranged in tabular form as follows. The number heading each column represents k, and the upper and lower numbers of the pairs in the columns represent the corresponding values of x_k and x_k' respectively.

$k=$	0	1	2	3	4	5	6	7	8	9	10
$p=67$ $x_k=$	60	30	15	41	54	27	47	57	62	31	49
$b=7$ $x_k'=$	7	37	52	26	13	40	20	10	5	36	18
$p=16231$ $x_k=$	16104	8052	4026	2013	9122	4561	10396	5198	2599	9415	12823
$b=127$ $x_k'=$	127	8179	12205	14218	7109	11670	5835	11033	13632	6816	3408
$p=431$ $x_k=$	430	215	323	377	404	202	101	266	133	282	141
$b=1$ $x_k'=$	1	216	108	54	27	229	330	165	298	149	290

In the above table, the numbers in the rows x_k or the rows x_k' that are obtained by division by 2 are sufficiently evident. As illustrations of the application of (L) and (L'), we have, for $p=67$ and $b=7$, $x_1'=7+30=37$; $x_2'=37+15=52$; $x_3=15+26=41$; $x_4=41+13=54$; etc. For $p=16231$ and $b=127$, $x_6=4561+5835=10396$; $x_7'=5835+5198=11033$; etc.

As illustrated in the above cases, we always have, when $a=2$ and b is numerically less than p, $x_1=(p-b)/2$ and $x_1'=(p+b)/2$,

In the manner indicated one may, after a little practice, compute a table of the kind indicated above almost as rapidly as the numbers it contains can be written down. For the single division by 2, or else the single addition, by which each of the numbers that are to be entered in the table are obtained, may be performed mentally as the numbers are recorded. Furthermore, when x_k and x_k' are calculated by this means, the formula (K) yields an independent and effective check, and one that can be applied at a glance. When the ar-

rangement of the table is as above, for example, the sum of the two numbers in each rectangle gives the corresponding prime. However, it is unnecessary to check each pair, x_k, x'_k, but it is sufficient to apply the test at intervals of a half score terms or more. If ordinary care is exercised, a table thus computed and checked needs no further verification.

7. A table of values of x_k and x'_k for $a = 2$.* The author has undertaken the elaboration of a table of values of x_k and x'_k for $a = 2$, $b = 1$, $k = 1, 2, 3, \ldots 100$, and for all primes under 100000. This table is now under way. A table of x_k alone for $k = 1, 2, \cdots 19$ and for all primes under 25000 was computed† by the method of §3, but the formation of the larger table for both x_k and x'_k was subsequently begun anew by the more rapid method of §6. The arrangement of the table is that shown in §6, except that the first column is headed p and contains in order the successive primes 3, 5, 7, 11, $\cdots$ and the column for $k = 0$ is omitted.

The table, when completed, will be applicable to the determination of primes and the formation of factor-tables of numbers of the forms $n2^k \pm 1$ up to 10^{10}, and to the determination of all factors under 100000 of numbers of these forms beyond 10^{10}, for $k \leqq 100$.

8. An application of the table. We pass now to the application of the table to the determination of the primes in the two arithmetical series:

$$(2) \qquad 2^{19} + 1, \quad 2(2^{19}) + 1, \quad 3(2^{19}) + 1, \cdots \quad 1211(2^{19}) + 1,$$

$$(2') \quad -2^{19} + 1, \quad -2(2^{19}) + 1, \quad -3(2^{19}) + 1, \cdots -1211(2^{19}) + 1.$$

If we include 1 with the numbers in (2) and (2'), we may write instead of the two series the single arithmetical series:

$$(2'') \quad -1211(2^{19}) + 1, \quad -1210(2^{19}) + 1, \cdots$$
$$-2^{19} + 1, \quad 1, \quad 2^{19} + 1, \quad 2(2^{19}) + 1, \cdots 1211(2^{19}) + 1.$$

Writing down in order the coefficients of 2^{19}, we obtain

$$(II) \quad -1211, \quad -1210, \quad -1209, \cdots$$
$$-3, \quad -2, \quad -1, \quad 0, \quad 1, \quad 2, \quad 3, \cdots 1209, \quad 1210, \quad 1211.$$

* A paper of Mr. L. L. Dines, to be published in the April number of the ANNALS, contains a detailed discussion of the case $a = 6$ and $b = 1$, with various applications and the description of a table he has elaborated.

† By the author and an assistant, Miss C. Ames, in April, 1905. This table is included in the appendix of the author's thesis on "Fermat's numbers and numbers of the forms $2^q q \pm 1$" on file in the Yale University Library.

Referring to the column headed 19 in the table, we find

$$x_{19,3} = 1, \quad x_{19,5} = 3, \quad x_{19,7} = 3, \quad x_{19,11} = 9, \cdots$$

Striking out from the numbers in (II), the positive and negative multiples of 3, increased by 1, —namely, 1, 4, 7, 10, 13, · · · 1201, 1204, 1207, 1210; − 2, − 5, − 8, − 11, − 14, · · · − 1202, − 1205, − 1208, − 1211; then from the numbers remaining in (II) the multiples of 5, increased by 3, —namely, 3, 8, 18, · · · 1203, 1208; − 7, − 22, · · · − 1207; then from those remaining the multiples of 7, increased by 3, —namely, 17, 24, · · · 1200; − 4, − 18, · · · − 1194; and so continuing, for all primes under $\sqrt{\{1211(2^{19})+1\}}$, the numbers finally remaining in (II), except 0, are the coefficients of the prime numbers that occur in the series $(2'')$, or, which amounts to the same, in (2) and $(2')$.

The author has carried out the work for this series as indicated, and has found that the following numbers in (II), and these only, correspond to primes in $(2'')$:

− 1198,	− 1195,	− 1186,	− 1164,	− 1156,	− 1143,	− 1141,	− 1134,	− 1129,	− 1126,	− 1125,	− 1123
− 1108,	− 1104,	− 1095,	− 1090,	− 1084,	− 1083,	− 1060,	− 1045,	− 1044,	− 1039,	− 1030,	− 1021
− 978,	− 973,	− 958,	± 936,	− 925,	− 924,	− 919,	− 910,	± 894,	− 888,	− 885,	− 883
− 868,	− 853,	− 850,	− 828,	− 804,	− 801,	− 793,	− 778,	− 759,	− 754,	− 748,	− 745
− 735,	− 733,	− 730,	− 726,	− 705,	− 700,	− 696,	− 685,	− 664,	− 663,	− 654,	± 609
− 603,	− 601,	− 600,	− 595,	− 579,	− 556,	− 540,	− 534,	− 520,	− 493,	− 484,	− 463
− 460,	− 456,	− 444,	− 441,	− 421,	− 415,	− 414,	± 411,	− 408,	− 400,	− 394,	− 391
± 369,	− 363,	− 360,	− 339,	± 324,	− 315,	− 313,	− 306,	− 303,	− 285,	− 268,	− 265
± 264,	− 241,	− 234,	− 229,	− 223,	− 213,	− 208,	− 199,	− 153,	− 150,	− 139,	− 133
− 115,	− 114,	− 103,	− 99,	− 93,	− 91,	− 76,	− 75,	− 73,	− 70,	± 69,	− 61
− 51,	− 36,	− 34,	− 1,								
11,	14,	26,	27,	41,	44,	50,	54,	57,	± 69,	71,	77
131,	132,	134,	147,	156,	159,	179,	191,	194,	197,	200,	212
216,	221,	225,	230,	252,	260,	± 264,	270,	282,	296,	302,	309
312,	320,	± 324,	335,	341,	356,	365,	± 369,	380,	390,	404,	± 411
417,	419,	422,	435,	440,	447,	459,	467,	470,	476,	485,	489
492,	497,	516,	524,	551,	554,	555,	582,	585,	594,	± 609,	627
629,	630,	641,	662,	671,	686,	690,	714,	719,	720,	732,	737
740,	747,	755,	762,	771,	782,	792,	795,	809,	810,	819,	827
837,	854,	875,	876,	879,	884,	± 894,	896,	929,	± 936,	942,	945
951,	987,	989,	1001,	1014,	1019,	1031,	1049,	1059,	1077,	1089,	1094
1100,	1110,	1115,	1127,	1136,	1140,	1154,	1155,	1161,	1167,	1169,	1181
1185.											

The numbers above that are marked with a double sign, as ± 936, occur in both the negative and positive coefficients. Whenever such is the case,

the corresponding numbers in $(2'')$, as $\pm 936(2^{19}) + 1$, differ numerically by 2, and compose what is called a "prime pair." There are eight such pairs occurring in $(2'')$.

After the table had been made, the time occupied in applying it to the determination of the primes occurring in (2) and $(2')$ was about eight hours. This is a shorter period than would be required in multiplying out the 2422 numbers in (2) and $(2')$ and consulting factor-tables like those of Burckhardt, Dase and Glaisher, if such factor-tables were extended as far as 634912769, the largest number considered. All the numbers in (2) and $(2')$ in which the coefficient of 2^{19} is numerically greater than 19 lie beyond 10^7 and hence beyond the range of the factor-tables.

In the above case, while the numbers in $(2'')$ range in numerical value from 524287 to 634912769, the "sifting" process is applied simply to the coefficients of 2^{19}, a set of consecutive integers ranging from -1211 to $+1211$. The power of the general method of this paper is due to just this feature. After a table of x_{kp} of sufficient extent to meet the requirements of the case in hand has been computed, the sifting is applied not directly to the numbers under investigation, but to a range of smaller consecutive integers, the coefficients of a^k. Thus is obtained a method, not difficult to apply, for the determination of the primes and the factorization of the composite numbers of a given form $xa^k + b$, where the investigation of each number separately would be quite impracticable.

9. Application to the factors of Fermat's Numbers. As is well known, all factors of Fermat's Numbers,* $F_n = 2^{2^n} + 1$, must be of the form

$$Q_n = q \cdot 2^{n+2} + 1.$$

All the known factors of F_n were identified by testing the divisibility of F_n by primes of the form Q_n. The primes of the form $q.2^{19} + 1$, given in §8, were tested as possible Fermat factors† for $q = 11, 14, 26, \cdots 989$, with the result that no Fermat factors were found in addition to those already known. Hence, for $n > 16$, there are no more factors of F_n in addition to those previously known, under $1001.2^{19} + 1$, or 524812289.

* F_n is known to be prime for $n = 0, 1, 2, 3, 4$, and composite for $n = 5, 6, 7, 9, 11, 12, 18, 23, 36, 38, 73$. See *Proceedings of the London Mathematical Society*, ser. 2, vol. 1, p. 175; vol. 3, p. xxi; vol. 5, p. 237; *Messenger of Mathematics*, vol. 37, p. 65; *Bulletin of the American Mathematical Society*, vol. 11, p. 343; vol. 12, p. 449.

† *I. e.*, factors of Fermat's numbers.

By making use of the 55^{th} and 75^{th} columns of the table described in §7, in its present extent, it was found that neither $5 \cdot 2^{55} + 1$ nor $5 \cdot 2^{75} + 1$ admit factors under 2000. Subsequent tests showed that both are prime, and that $5 \cdot 2^{75} + 1$ is a factor of F_{73}. The last result was obtained by the following method.

Setting $p = 5 \cdot 2^{75} + 1$, we have

$$5 \cdot 2^{75} + 1 \equiv 0 \bmod p,$$

and hence, using the notation of §4,

$$\xi_{64} = x_{64\,p21} = 5 \cdot 2^{11}.$$

Then applying (H) we obtain successively

$$\xi_{27} \equiv -\xi_{64}^2 \equiv -5^2 \cdot 2^{22} \bmod p$$
$$\xi_{29} \equiv -5^4 \cdot 2^{44}$$
$$\xi_{29} \equiv -5^8 \cdot 2^{88} \equiv 5^7 \cdot 2^{13}$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\xi_{212} \equiv 5^{26} \cdot 10^{29}.$$

From this stage on, the numerically least remainder (mod p) of each $\xi_{2^n}^2$ was found by direct division by p. This process showed finally $\xi_{2^{73}} = 1$, and therefore that $2^{75} + 1$ is divisible by p. Thus the identification of the Fermat factor $5.2^{75} + 1$ was effected solely by the methods of the present paper,* and less arduously, as far as ξ_{212}, than by the usual method of calculating successively the remainders (mod p) of $2^{2^7}, 2^{2^8}, \ldots 2^{2^{73}}$.

A more detailed report on the application of the methods of this paper to Fermat's factors will be held, for the sake of completeness, until the table of §7 is further advanced.

10. Connection with Lebon's table. Professor Lebon, in a paper published in the *Bulletin of the American Mathematical Society*, vol. 13, p. 74, describes a process for the construction of a table applicable to the factorization of the numbers in arithmetical progressions in which the common difference is the simple product of consecutive primes, $2.3.5.\cdots p_\lambda$, and the first term is any one of the $\phi(2.3.5.\cdots p_\lambda)$ numbers less than and

* This method is stated briefly in different form in *Bulletin of the American Mathematical Society*, vol. 12, p. 237.

prime to the common difference. The common difference and first term, called the *base* and *index* of the progression, are represented by B and I respectively. The system of $\phi(B)$ arithmetical progressions whese general term is $BK + I$, where K is successively equal to the positive integers starting from zero, and I is prime to B, contains all prime numbers except $2, 3, 5, \cdots p_\lambda$, and all composite numbers except multiples of $2, 3, 5, \cdots p_\lambda$. The numbers recorded in the table are the minimum values of K that render $BK + I$ divisible by each successive prime following p_λ.* Evidently a table so constructed is applicable to the decomposition of, or the determination of the primality of successive numbers in progressions of this special form, as far as p_ν^2 (the last prime included in the table), or to the investigation of the divisibility of isolated numbers in such progressions, in a manner similar to that described in the present paper. In fact, if we take $a = B$, $b = \mathrm{I}$ and $k = 1$, the numbers x_1 occurring in my table in the first column following the primes coincide, for the primes following p_λ, with the minimum values of K recorded in Professor Lebon's table. If the latter table is extended to include the $\phi(B)$ values of I, the complete table is clearly applicable to *successive* integers, aside from multiples of $2, 3, \cdots p_\lambda$, and hence applicable to the formation of factor-tables of the usual form.

However, the method of Professor Lebon is not so easily and directly applicable as is the method of the present paper to the investigation of the divisibility of numbers of the form $ma^k + b$ when k exceeds unity. Thus the special form of the numbers $m.2^k + 1$, treated in §§8, 9, would be of no advantage if the table of Professor Lebon were to be applied.

11. Application to ranges of consecutive integers. The method of this paper may be extended so as to be applicable to a range of consecutive integers without restricting the form of the progressions as in §10. Let a and k be fixed. Then, like the Lebon progressions, the system of $\phi(a^k)$ arithmetical progressions represented by $ma^k + b$, where $m = 0, 1, 2, \cdots$ and b assumes successively the $\phi(a^k)$ positive integral values less than and prime to a^k, contains all positive integers that are prime to a. Consequently, a table of values of x_{kpab} for the primes as far as π and for the $\phi(a^k)$ values of b would suffice, like the Lebon table, for the decomposition, or the determination of the primality of all numbers prime to a as far as π^2.

* A table compiled by Professor Lebon has been published under the title *Table de caractéristiques relatives à la base* 2310 *des facteurs premiers d'un nombre inférior à* 30030, Paris, 1906.

The efficiency of such a table, as well as the amount of labor involved in the compilation, will be determined by our choice of a and k. To fix ideas, suppose it is desired to construct a table effective to 10^8. A convenient choice would be $a = 10$, $k = 4$. Then the system of $\phi(10^4)$, or 4000, progressions of 10000 terms each, whose general term is $m.10^4 + b$, contains every odd number except multiples of 5, from 1 to $10^8 - 1$. To be effective as far as 10^8, the corresponding table of values of $x_{4p\,10b}$ should be calculated for 1226 primes, viz., 7, 11, 13, $\cdots$ 9973.* The following is a convenient arrangement of the table. Here $x_{4p\,10b}$ stands at the intersection of the row p with the column b.

$a = 10$, $k = 4$.

$b =$	1	3	7	9	. . .	233	237	
$p = 7$	5	1	0	3	. . .	3	2	. . .
11	10	8	4	2	. . .	9	5	. . .
. .	. .	. .	. .	. .	. . .	. .	. .	. .
997	432	299	33	897	. . .	956	690	. . .
. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .	. . .

To illustrate the method of applying the table, suppose we wish to decompose the number 26840237, $= N$, say. It is easily seen that 3 is not a factor of N. Writing $N = 2684 \times 10^4 + 237$, and referring to the column headed 237, we find

$$2684 \not\equiv 2 \bmod 7$$
$$\not\equiv 5 \bmod 11$$
$$\cdots\cdots$$
$$\equiv 690 \bmod 997.$$

Here 690 is the *first* remainder obtained on dividing 2684 by a prime that coincides with the corresponding x_{pb}. Hence 997 is the *least* factor of N. The remaining factor, 26921, is less than 997^2 and therefore prime. Thus we have the complete decomposition, $N = 997 \times 26921$. If for the primes as far

* It is useless to include 3, as divisibility by 3 is easily tested.

as $\sqrt{N}$ no remainder were found to coincide with the corresponding x_{pb}, then would N be prime.

The application of the table would be greatly facilitated if a residue table were available, giving the least positive remainders of each integer as far as 10^4 for the prime moduli 7, 11, $\cdots$ 9973. If the arrangement of the residue table were similar to that of the above table, then to effect the decomposition of the number 26840237, for example, we should need only to compare corresponding numbers in the columns headed 2684 and 237 in the residue and factorization tables respectively, until a coincidence is discovered at $p = 997$.

The application of the above table has one evident advantage over that of Professor Lebon, covering the same range of numbers. We recognize at sight the coefficients required to express a number in the form $m.10^4 + b$; while a division must be performed to express a number in the form $m(2 \cdot 3. \cdots p_\lambda) + b$ preliminary to applying the Lebon table. Aside from this feature, the two tables differ little in efficiency when a residue table is available with each.

The labor of compilation is not so great as to render the task impracticable. Starting from the value of x_0 given by (C), §3, we calculate successively, for given values of b and p, x_1, x_2, x_3 and finally x_4 by the method of §3. Aided by the use of calculating machines, a computer of average speed can evaluate by this method and record about one hundred and twenty x_4's per hour, when the corresponding primes contain not more than four places. The labor may be further reduced by means of a relation which we shall now derive.

Let us write, for brevity, x_b for $x_{4p\,10b}$. We have, by definition,

$$10^4 \cdot x_b + b \equiv 0 \bmod p$$

and

$$10^4 \cdot x_{b'} + b' \equiv 0 \bmod p.$$

Whence

$$10^4(x_b + x_{b'}) + b + b' \equiv \bmod p,$$

and therefore

$$(M) \qquad x_{b+b'} \equiv x_b + x_{b'} \bmod p.$$

When the numbers in column 1 have been found by the method of §3, repeated applications of (M) will give all the remaining columns. However, the relation (M) can be used most effectively if two auxiliary columns have

been calculated, viz., for $b = 2, 4$.* Since two successive values of b in the table differ by 2 or 4, the addition of x_2 or x_4 to any x_b and the subtraction of p if the sum exceeds $p - 1$ will give the corresponding x_{b+2} or x_{b+4} in the next column. Thus for $p = 997$, we have $x_2 = 864$, $x_4 = 731$; $x_9 = x_7 + x_2 = 33 + 864 = 897$; $x_{237} \equiv x_{233} + x_4 \equiv 956 + 731 \equiv 690 \bmod p$.

If a calculating machine is used, a computer can evaluate by means of (M) and record from four to five hundred x_b's per hour, for primes of four places or less. The table may be checked by applying the method of §3 to every tenth or twelfth column.† We may estimate, therefore, that a single computer would require four to five years for calculating, recording and checking the x_b's for the entire table of 4000 columns and 1226 primes.‡

The generalization of the relations developed in this paper for composite moduli, and the incidental application of the methods of deriving x_{kpab} and x'_{kpab} to the calculation of power residues are only remotely connected with the purpose of this paper and will be reserved for later publication.

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APRIL, 1908.

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* Here b is not prime to a, so that these two columns should not be included in the table. It is easily seen that (M) is valid for $b = 2, 4$.

† It is evident from the definitions that $x_b = x_{b+p}$. Hence the numbers in any row of the table recur in exact order at intervals of $10p$ in the values of b. For primes under 1000 this property facilitates the calculation to some extent.

‡ The manuscript factor-tables of Professor Kulik, left in charge of the Vienna Academy, extend to 100330201, — a little beyond the proposed limit of the factorization table here described. However, the Kulik tables are at present almost useless on account of the many errors they contain. See *Bulletin of the American Mathematical Society*, vol. 14, p. 106.

A METHOD OF INVESTIGATING NUMBERS OF THE FORMS $6^k \cdot s \pm 1$.*

By Lloyd L. Dines

In a paper published in the January issue of the Annals, Dr. Morehead indicated a method for determining the factors of numbers of the form $a^k \cdot s + b$ where a, k and s are positive integers and b is a positive or negative integer. He elaborated the case when $a = 2$, $b = \pm 1$, and considered briefly the case when $a = 10$. The present paper will describe in some detail the method as applied to the case when $a = 6$, $b = \pm 1$, and will exhibit some results obtained.

1. The Functions x_k and x_k'. Let us consider the congruences

$$6^k \cdot x - 1 \equiv 0 \bmod p, \tag{1}$$

$$6^k \cdot x + 1 \equiv 0 \bmod p, \tag{2}$$

when k is any positive integer and p is any prime greater than 3. [It is evident that (1) and (2) would have no solution for $p = 2$ or 3.] Solutions of each of these congruences exist, and are functions of p and k. For any particular value of p, we shall call the least positive solution of (1) x_k, and the least positive solution of (2) x_k'. From the linear character of the congruences (1) and (2), it follows that the functions x_k and x_k' are uniquely determined, each being less than p.

These functions possess a number of interesting properties, any or all of which may be used as a basis of computation of the functions. We will state these properties without proof, merely making reference at the right-hand side of the page to the general theorems stated and proven in Dr. Morehead's paper.

2. Properties of x_k and x_k'.

1°)	$x_k + x_k' = p.$	(§5)
2°)	$x_k \cdot x_r \equiv x_{k+r} \bmod p.$	(§4, H')
3°)	$x_k' \cdot x_r \equiv x_{k+r}' \bmod p.$	(§4, H')
4°)	$x_k' \cdot x_r' \equiv x_{k+r} \bmod p.$	(§4, H)

* Read before the Chicago Section of the American Mathematical Society, April 17, 1908.

5°) *When p is of the form $6n - 1$,*

(*a*) $x_{k+1} \equiv nx_k \bmod p$, (§4, *F*)

(*b*) $x'_{k+1} \equiv nx'_k \bmod p$.

6°) *When p is of the form $6n + 1$,*

(*a*) $x_{k+1} \equiv nx'_k \bmod p$, (§4, *F'*)

(*b*) $x'_{k+1} \equiv nx_k \bmod p$.

3. Method of Computing x_k and x'_k. For any fixed value of p, and for $k = 1$, the values of x_k and x'_k are easily determined by the following method. We have identically $6\left(\frac{p+1}{6}\right) \equiv 1 \bmod p$.

Therefore $x_1 \equiv \frac{p+1}{6} \bmod p$, using the ordinary definition of fractional congruences when $p + 1$ is not divisible by 6. Similarly $6\left(\frac{p-1}{6}\right) \equiv -1 \bmod p$, and therefore $x'_1 \equiv \frac{p-1}{6} \bmod p$. Either $\frac{p+1}{6}$ or $\frac{p-1}{6}$ is integral. In the former case $x_1 = \frac{p+1}{6}$, and from property 1°, $x'_1 = p - \frac{p+1}{6}$. In the latter case $x'_1 = \frac{p-1}{6}$ and $x_1 = p - \frac{p-1}{6}$.

After x_1 and x'_1 have been thus determined, the values of x_k and x'_k for consecutive values of k can be computed by means of the following recursion formulas,* in each of which n has some one of the values 0, 1, 2, 3, 4, 5 :—

$$(A) \quad x_{k+1} = \frac{x_k + np}{6},$$

$$(B) \quad x'_{k+1} = \frac{x'_k + np}{6}.$$

The determination in each case of the value of n necessary to make $\frac{x_k + np}{6}$ an integer less than p, is facilitated by the following simple *Theorem:* If x_k is odd, n will be 1, 3, or 5. If x_k is even, n will be 2 or 4.

For, in order to be divisible by 6, $x_k + np$ must be even. If x_k is odd but

* Cf. ANNALS OF MATHEMATICS, vol. 10, p. 92 (1909).

$x_k + np$ is even, then must np also be odd, and hence n must be odd, i. e., 1, 3, or 5. If x_k is even, then must np also be even, and hence n must be even, i. e., 2 or 4.

Formulas (A) and (B) afford the easiest method of obtaining the functions x_k and x'_k, while the properties of section 2 can be very profitably used for checking the computation. The following table will show the functions x_k and x'_k of four consecutive primes for values of k from 1 to 12:

Prime	k	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$	$k=9$	$k=10$	$k=11$	$k=12$
157	x_k	131	48	8	106	70	64	63	89	41	33	84	14
	x'_k	26	109	149	51	87	93	94	68	116	124	73	143
163	x_k	136	77	40	61	146	133	158	135	104	126	21	85
	x'_k	27	86	123	102	17	30	5	28	59	37	142	78
167	x_k	28	116	75	96	16	114	19	31	33	89	154	137
	x'_k	139	51	92	71	151	53	148	136	134	78	13	30
173	x_k	29	149	169	57	96	16	118	135	109	47	152	83
	x'_k	144	24	4	116	77	157	55	38	64	126	21	90

4. Quadratic Characters of x_k and x'_k. In the following section we shall show how these functions x_k and x'_k can be paired off into two sets, such that one set consists of some or all of the quadratic residues, while the other set consists of some or all of the quadratic non-residues of any prime greater than 3.

1. *Quadratic Character of x'_k.*

We have by definition, $6^k x'_k + 1 \equiv 0 \bmod p$. Hence

$$x'_k \equiv -\frac{1}{6^k} \bmod p,$$

and

$$\left(\frac{x'_k}{p}\right) = \left(\frac{-\frac{1}{6^k}}{p}\right) = \left(\frac{-6^k}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{2}{p}\right)^k\left(\frac{3}{p}\right)^k. \qquad (3)$$

From the Theory of Quadratic Residues, we know

$$\left(\frac{-1}{p}\right) = \begin{cases} +1 & \text{when } p = 4m+1, \\ -1 & \text{when } p = 4m+3; \end{cases}$$

$$\left(\frac{2}{p}\right) = \begin{cases} +1 & \text{when } p = 8m \pm 1, \\ -1 & \text{when } p = 8m \pm 3; \end{cases}$$

$$\left(\frac{3}{p}\right) = \begin{cases} +1 & \text{when } p = 12m \pm 1, \\ -1 & \text{when } p = 12m \pm 5. \end{cases}$$

From these formulas and equations (3), we obtain the followng results:

When $p = \begin{Bmatrix} 24m+1 \\ \text{or} \\ 24m+5 \end{Bmatrix}$, $\left(\frac{x_k'}{p}\right) = (+1)(\pm 1)^k(\pm 1)^k = +1.$

When $p = \begin{Bmatrix} 24m-1 \\ \text{or} \\ 24m-5 \end{Bmatrix}$, $\left(\frac{x_k'}{p}\right) = (-1)(\pm 1)^k(\pm 1)^k = -1.$

When $p = \begin{Bmatrix} 24m+7 \\ \text{or} \\ 24m+11 \end{Bmatrix}$, $\left(\frac{x_k'}{p}\right) = (-1)(\pm 1)^k(\mp 1)^k = \begin{cases} +1 \text{ for } k \text{ odd.} \\ -1 \text{ for } k \text{ even.} \end{cases}$

When $p = \begin{Bmatrix} 24m-7 \\ \text{or} \\ 24m-11 \end{Bmatrix}$, $\left(\frac{x_k'}{p}\right) = (+1)(\pm 1)^k(\mp 1)^k = \begin{cases} +1 \text{ for } k \text{ even.} \\ -1 \text{ for } k \text{ odd.} \end{cases}$

2. *Quadratic Character of x_k.*

We have by definition, $6^k x_k \equiv 1 \bmod p$. Hence

$$x_k \equiv \frac{1}{6^k} \bmod p,$$

and

$$\left(\frac{x_k}{p}\right) = \left(\frac{\frac{1}{6^k}}{p}\right) = \left(\frac{6^k}{p}\right) = \left(\frac{2}{p}\right)^k \left(\frac{3}{p}\right)^k. \tag{4}$$

From (4) we get the following results :—

When $p = \left\{\begin{matrix} 24m+1 \\ \text{or} \\ 24m+5 \end{matrix}\right\}$, $\left(\frac{x_k}{p}\right) = (\pm 1)^k(\pm 1)^k = +1.$

When $p = \left\{\begin{matrix} 24m-1 \\ \text{or} \\ 24m-5 \end{matrix}\right\}$, $\left(\frac{x_k}{p}\right) = (\pm 1)^k(\pm 1)^k = +1.$

When $p = \left\{\begin{matrix} 24m+7 \\ \text{or} \\ 24m+11 \end{matrix}\right\}$, $\left(\frac{x_k}{p}\right) = (\pm 1)^k(\mp 1)^k = \begin{cases} +1 \text{ for } k \text{ even.} \\ -1 \text{ for } k \text{ odd.} \end{cases}$

When $p = \left\{\begin{matrix} 24m-7 \\ \text{or} \\ 24m-11 \end{matrix}\right\}$, $\left(\frac{x_k}{p}\right) = (\pm 1)^k(\mp 1)^k = \begin{cases} +1 \text{ for } k \text{ even.} \\ -1 \text{ for } k \text{ odd.} \end{cases}$

We can summarize these results in the following convenient form :—

(*a*) When $p = \left\{\begin{matrix} 24m+1 \\ \text{or} \\ 24m+5 \end{matrix}\right\}$, $\left(\frac{x'_k}{p}\right) = \left(\frac{x_k}{p}\right) = +1.$

(*b*) When $p = 24m-1$, $\left(\frac{x'_k}{p}\right) = \left(\frac{x_k}{p}\right) = -1.$

(*c*) When $p = 24m-5$, $\left(\frac{x'_k}{p}\right) = -1$, $\left(\frac{x_k}{p}\right) = +1.$

(*d*) When $p = \left\{\begin{matrix} 24m+7 \\ \text{or} \\ 24m+11 \end{matrix}\right\}$, $\left(\frac{x'_k}{p}\right) = \begin{cases} +1 \text{ for } k \text{ odd.} \\ -1 \text{ for } k \text{ even.} \end{cases}$

$\left(\frac{x_k}{p}\right) = \begin{cases} +1 \text{ for } k \text{ even.} \\ -1 \text{ for } k \text{ odd.} \end{cases}$

(*e*) When $p = \left\{\begin{matrix} 24m-7 \\ \text{or} \\ 24m-11 \end{matrix}\right\}$, $\left(\frac{x'_k}{p}\right) = \left(\frac{x_k}{p}\right) = \begin{cases} +1 \text{ for } k \text{ even.} \\ -1 \text{ for } k \text{ odd.} \end{cases}$

As an illustration of the application of these results, we will find all the quadratic residues and non-residues of 67.

The values of x_k and x_k' for the prime 67 and for $k = 1, 2, \cdots 33$, are:

k	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
x_k'	11	13	58	32	50	53	20	48	8	46	30	5	12	2	45	41	18
x_k	56	54	9	35	17	14	47	19	59	21	37	62	55	65	22	26	49

k	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33
x_k'	3	34	28	27	38	51	42	7	57	43	63	44	52	31	61	66
x_k	64	33	39	40	29	16	25	60	10	24	4	23	15	36	6	1

Now $67 = 24m - 5$. Therefore from (c), the quadratic residues are:

56, 54, 9, 35, 17, 14, 47, 19, 59, 21, 37, 62, 55, 65, 22, 26, 49, 64, 33, 39, 40, 29, 16, 25, 60, 10, 24, 4, 23, 15, 36, 6, 1.

The quadratic non-residues are:

11, 13, 58, 32, 50, 53, 20, 48, 8, 46, 30, 5, 12, 2, 45, 41, 18, 3, 34, 28, 27, 38, 51, 42, 7, 57, 43, 63, 44, 52, 31, 61, 66.

An important use of the residues and non-residues obtained in the above manner is found in their application to the factorization of single numbers of the form $6^k s \pm 1$.

5. Application of Functions x_k and x_k' to the Factorization of "Ranges" of Numbers of the Forms $6^k s \pm 1$. Consider any number of the form $6^k s + 1$. The necessary and sufficient condition that this number be composite is that there be a number p, such that the congruence $6^k s + 1 \equiv 0 \bmod p$ be satisfied. In other words, the problem of factoring this number reduces to the problem of finding a prime number p, less than the square root of $6^k s + 1$, such that the x_k' function of this number is either equal to, or congruent to s, modulo p. In the same way, to factor any number $6^k s - 1$, it is necessary and sufficient to find a prime number p less than the square root of $6^k s - 1$, such that its x_k function is equal to, or congruent to s, modulus p. If such a number does not exist, in either the first case or the second the number under consideration is prime.

A table* has been computed by the author, which shows the functions x_k and x_k' of all primes from 5 up to 10,000, for $k = 1, 2, 3 \cdots 12$. By a table with such limits, it is possible to investigate completely all numbers of the forms $6^k s \pm 1$ ($k \leqq 12$) up to 100,000,000 and to obtain all factors under 10,000 of numbers of these forms over 100,000,000. The power of the table is better realized when we consider that we now have no factor tables over 10,000,000.

The table is most profitably used to investigate "ranges" of numbers of the forms $6^k s \pm 1$ for any *fixed* value of k and for a series of consecutive values of s. The method used is entirely analagous to the method of the Sieve of Eratosthenes for determining prime numbers in general.

We first write down the values of s between the limits considered. Then, beginning at the first of the table, we look in the column of the particular value of k which is being investigated. If we are considering the form $6^k s - 1$, we look only at the x_k functions. If our form is $6^k s + 1$, we look only at the x_k' functions. If the x_k (or x_k') for the first prime, 5, is among the values of s which we have written down, we cross it out, together with all numbers congruent to it modulo 5. All numbers with these values of s are divisible by 5, because by the definition of x_k (or x_k'), this value of s will satisfy the congruence $6^k s \mp 1 \equiv 0 \bmod 5$. Then if the x_k (or x_k') of the next prime, 7, occurs among the values of s, we cross it out together with all congruent numbers modulo 7. All numbers having these values of s are divisible by 7. Continuing in this way, we cross out all the remaining values of s which occur as x_k (or x_k') functions together with numbers which are congruent for the corresponding modulus. All numbers having these values of s are divisible by the corresponding modulus p. This process need be carried on only until p reaches the value $\sqrt{6^k s \mp 1}$, where s has its maximum value in the "range," since a number is prime if it has no factor less than its square root. For all values of s remaining after this process, $6^k s - 1$ (or $6^k s + 1$) is prime.

We will now give a few results obtained in some "ranges" by this method.

Range 1. $6^6 s + 1$. $s = 200, 201, \cdots 300$, and $2150, 2151, \cdots 2250$.

The numbers are prime for the following values of s: 207, 213, 215, 225, 236, 243, 270, 282, 295.

* In manuscript at the Library of Northwestern University.

The numbers corresponding to the following values of s have no factors under 10,000:

2150, 2165, 2166, 2170, 2171, 2175, 2190, 2203, 2225, 2226, 2227, 2241, 2242, 2247.

These numbers lie between 9,331,201, and 104,976,001.

Range 2. $6^6s - 1$. $s = 200, 201, \cdots 300$, and $2150, 2151, \cdots 2250$.

The numbers are prime for the following values of s: 200, 205, 214, 217, 224, 234, 238, 239, 269, 288, 293, 294.

The numbers corresponding to the following values of s have no factors under 10,000:

2163, 2168, 2172, 2180, 2184, 2189, 2203, 2207, 2212, 2214, 2217, 2240, 2250.

These numbers lie between 9,331,199, and 104,975,999.

Range 3. $6^7s + 1$. $s = 1, 2, \cdots 400$.

The numbers are prime for the following values of s:

3, 10, 17, 18, 25, 33, 45, 47, 52, 53, 60, 62, 75, 83, 86, 100, 110, 112, 121, 126, 131, 133, 138, 140, 145, 150, 163, 165, 170, 171, 172, 182, 186, 187, 200, 215, 226, 237, 241, 242, 255, 256, 272, 273, 276, 280, 283, 286, 293, 300, 303, 307, 313, 335, 341, 347, 348, 353.

The numbers corresponding to the following values of s have no factors under 10,000:

361, 371, 385, 391, 396, 398.

These numbers lie between 279,937 and 111,974,401.

Range 4. $6^7s - 1$. $s = 1, 2, \cdots 400$.

The numbers are prime for the following values of s:

5, 9, 33, 39, 49, 53, 64, 65, 75, 78, 82, 98, 105, 108, 124, 127, 133, 134, 142, 152, 163, 168, 175, 177, 179, 185, 187, 189, 199, 203, 213, 214, 219, 224, 225, 229, 233, 234, 239, 253, 255, 257, 259, 277, 278, 285, 287, 297, 298, 308, 310, 312, 319, 322, 333, 352, 357.

The numbers corresponding to the following values of s have no factors under 10,000:

362, 364, 369, 375, 383, 390, 394, 400.

These numbers lie between 279,935, and 111,974,399.

Range 5. $6^8 s + 1$. $s = 1, 2, \cdots 100$.

The numbers are prime for the following values of s:

3, 10, 21, 23, 25, 31, 46, 50, 58.

The numbers corresponding to the following values of s have no factors under 10,000:

66, 67, 72, 87, 93.

These numbers lie between 1,679,617 and 167,961,601.

Range 6. $6^8 s - 1$. $s = 1, 2, \cdots 100$.

The numbers are prime for the following values of s:

13, 18, 28, 39, 40, 55, 59.

The numbers corresponding to the following values of s have no factors under 10,000:

65, 70, 73, 79, 83, 95.

These numbers lie between 1,679,615, and 167,961,599.

Range 7. $6^9 s + 1$. $s = 1, 2, \cdots 100$.

The numbers corresponding to the following values of s have no factors under 10,000:

10, 11, 12, 26, 47, 48, 51, 53, 55, 62, 65, 68, 72, 73, 81, 87, 91, 97, 98, 100.

These numbers lie between 10,077,697, and 1,007,769,601.

Range 8. $6^9 s - 1$. $s = 1, 2, \cdots 100$.

The value $s = 3$ gives a prime.

The numbers represented by the following values of s have no factors under 10,000:

19, 35, 38, 43, 53, 70, 75, 80, 85, 89, 98.

These numbers lie between 10,077,695 and 1,007,769,599.

Range 9. $6^{10}s + 1$. $s = 1, 2, \cdots 100$.

The numbers represented by the following values of s have no factors under 10,000:

2, 8, 12, 18, 28, 31, 52, 53, 56, 58, 70, 71, 95.

These numbers lie between 60,466,177 and 6,046,617,601.

Range 10. $6^{10}s - 1$. $s = 1, 2, \cdots 100$.

The numbers represented by the following values of s have no factors under 10,000.

19, 24, 30, 37, 38, 44, 53, 58, 60, 68, 69, 70, 72, 87, 94.

These numbers lie between 60,466,175, and 6,046,617,599.

The numbers which have been designated as having no factors under 10,000 have been spoken of thus because the range of the table used is not great enough to determine absolutely that they are prime. Probably most of them are prime.

A number of very interesting observations can be made from the few results which we have tabulated. Notice, for instance, the consecutive values of s which give primes (or for those above 100,000,000 probably primes):—

Range 1: 2165, 2166; 2170, 2171; 2225, 2226, 2227; 2241, 2242.

Range 2: 238, 239; 293, 294.

Range 3: 17, 18; 52, 53: 170, 171, 172: 186, 187; 241, 242; 255, 256; 272, 273.

Range 4: 64, 65; 133, 134; 213, 214; 233, 234; 277, 278; 297, 298.

Range 5: 66, 67.

Range 6: 39, 40.

Range 7: 10, 11, 12; 47, 48; 72, 73; 97, 98.

Range 9: 52, 53; 70, 71.

Range 10: 37, 38; 68, 69, 70.

More interesting than these, perhaps, are the "prime pairs" (primes differing by 2), which are found in these ranges.

These occur, of course, whenever any value of s makes both $6^k s + 1$ and $6^k s - 1$ prime. The following "prime pairs" (those over 100,000,000 probably prime) are found in the above ranges :—

$6^6 \cdot 2203 \pm 1$	102,783,167	and	102,783,169
$6^7 \cdot 33 \pm 1$	9,237,887	and	9,237,889
$6^7 \cdot 53 \pm 1$	14,836,607	and	14,836,609
$6^7 \cdot 75 \pm 1$	20,995,199	and	20,995,201
$6^7 \cdot 133 \pm 1$	37,231,487	and	37,231,489
$6^7 \cdot 163 \pm 1$	45,629,567	and	45,629,569
$6^7 \cdot 187 \pm 1$	52,348,031	and	52,348,033
$6^7 \cdot 255 \pm 1$	71,383,679	and	71,383,681
$6^9 \cdot 53 \pm 1$	534,117,887	and	534,117,889
$6^9 \cdot 98 \pm 1$	987,614,207	and	987,614,209
$6^{10} \cdot 53 \pm 1$	3,204,707,327	and	3,204,707,329
$6^{10} \cdot 58 \pm 1$	3,507,038,207	and	3,507,038,209
$6^{10} \cdot 70 \pm 1$	4,232,632,319	and	4,232,632,321

The numbers of the first pair mentioned, i. e., "102, 783,167 and 102,783, 169" fall above one hundred million, the limit of the table above described, but they have been further tested and found to be prime. They are believed to constitute the largest prime pair known at present.

In conclusion we would call attention to the advantages of this method in determining prime numbers in general. Since all primes greater than 3 are of the forms $6s \pm 1$, if we take $k = 1$, in the method described above, and let s have consecutive values, all primes may be computed by making use of the first column of the table. The advantages of this over the ordinary "sieve" method are that the numbers to be handled are very much smaller, and only one third as many have to be examined.

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THE SOLUTION OF ALGEBRAIC EQUATIONS BY PARTIAL DIFFERENTIAL EQUATIONS

By H. A. Sayre

1. Let the algebraic equation of degree n in z be

$$z^n + nxz + ny = 0.$$

Let the solution of this equation be $z = F(x, y)$. With the usual notation for partial derivatives with respect to x and y, we have

$$p = -\frac{z}{z^{n-1} + x}, \qquad q = -\frac{1}{z^{n-1} + x},$$

$$r = \frac{2zx - (n-3)z^n}{(z^{n-1} + x)^3}, \qquad s = \frac{x - (n-2)z^{n-1}}{(z^{n-1} + x)^3},$$

$$t = -\frac{(n-1)z^{n-2}}{(z^{n-1} + x)^3},$$

$$a = \frac{(12n - 2n^2 - 16)z^{2n-1} + z^n x(n^2 + 6n - 19) - 6zx^2}{(z^{n-1} + x)^5},$$

$$\beta = \frac{(9n - 2n^2 - 9)z^{2n-2} + (n^2 + 3n - 8)z^{n-1}x - 2x^2}{(z^{n-1} + x)^5},$$

$$\gamma = \frac{(n-1)z^{n-2}\{(4 - 2n)z^{n-1} + (n+1)x\}}{(z^{n-1} + x)^5},$$

$$\delta = \frac{(n-1)(1-2n)z^{2n-4} + (n-1)(n-2)xz^{n-3}}{(z^{n-1} + x)^5}.$$

Among these derivatives there exist relations such as

$$p - qz = 0,$$

$$rt - s^2 - pqt + q^2 s = 0,$$

$$pz^{n-2} + qx + 1 = 0,$$

$$z = (n-1)\,px + nqy.$$

When n is assigned it may be possible to form a differential equation, into whose solution z and the arbitrary functions enter in such a way, that when the algebraic equation is considered as a particular solution, the integral equation reduces to

$$z - F(x, y) = 0,$$

where F is a known function. If so, the algebraic equation is solved.

2. Solution of the Quadratic. For $n = 2$, $pz^{n-2} + qx + 1 = 0$, reduces to

$$p + qx + 1 = 0.$$

Lagrange's subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{x} = \frac{dz}{-1}.$$

Two integrals are

$$x^2 - 2y = \text{constant}, \qquad z + x = \text{constant};$$

hence the general integral is

$$z = -x + \phi(x^2 - 2y).$$

ϕ must now be determined so as to make

$$z^2 + 2zx + 2y = 0$$

a particular solution.

1°. In the quadratic, when $x = 0$, we get

$$z = \pm\sqrt{-2y}, \text{ hence } \phi(-2y) = \pm\sqrt{-2y},$$

which determines the form of ϕ. The integral equation now reduces to

$$z = -x \pm \sqrt{x^2 - 2y},$$

which is the solution of the quadratic.

2°. Substituting $z = -x + \phi$ in the quadratic, we get $\phi^2 = x^2 - 2y$. This is the well known solution in which the second term is removed by a linear substitution.

3°. The integral equation suggests changing the variables from x and y

to x and v, where $v = x^2 - 2y$. The quadratic becomes

$$z^2 + 2zx + x^2 = v, \text{ hence } z + x = \pm\sqrt{v}.$$

This is the common method of completing the square.

Among the differential equations which may be used for solving the quadratic are the following:

1. $q = \dfrac{1}{\sqrt{x^2 - 2y}}$, $\qquad z = \psi(x) + \sqrt{x^2 - 2y}$.
2. $rx + sx^2 - p - 1 = 0$, $\qquad z = -x + \phi(x^2 - 2y) + \psi(y)$.
3. $s + tx = 0$, $\qquad z = \psi(x) + \phi(x^2 - 2y)$.
4. $\beta x + \gamma x^2 - s = 0$, $\qquad z = \phi(x) + \psi(x^2 - 2y) + \theta(y)$.

3. Solution of the Cubic. For the cubic we have $r + tx = 0$. When $n = 3$, $z = (n - 1)\, px + nqy$ becomes

$$z = 2px + 3qy,$$

whence

$$p = 2rx + 2v + 3sy.$$

Multiplying $r + tx = 0$ by x, and subtracting from the preceding equation, we get

$$rx + 3sy - tx^2 + p = 0.$$

To integrate this equation let us use Monge's method. The subsidiary equations are

$$\frac{dy}{dx} = \frac{3y \pm \sqrt{9y^2 + 4x^3}}{2x},$$

$$2d(px) - \frac{4x^3}{3y \pm \sqrt{9y^2 + 4x^3}}\, dq = 0,$$

from which we have

$$-3y + \sqrt{9y^2 + 4x^3} = c_1,$$

$$2px - (-3y + \sqrt{9y^2 + 4x^3})q = c_2,$$

$$-3y - \sqrt{9y^2 + 4x^3} = c_3,$$

$$2px - (-3y - \sqrt{9y^2 + 4x^3})q = c_4.$$

The intermediate integrals are

$$2px - (-3y + \sqrt{9y^2 + 4x^3})\, q = \theta_1 (-3y + \sqrt{9y^2 + 4x^3}),$$

$$2px - (-3y - \sqrt{9y^2 + 4x^3})\, q = \theta_2 (-3y - \sqrt{9y^2 + 4x^3}).$$

Putting $u = -3y$, $v = \sqrt{9y^2 + 4x^3}$, we get

$$dz = \frac{\{[u+v]\,\theta_2(u-v) - [u-v]\,\theta_1(u+v)\}}{4xv}\,dx + \frac{\theta_2(u-v) - \theta_1(u+v)}{2v}\,dy,$$

or

$$dz = \frac{\{[u+v]\,\theta_2(u-v) - (u-v)\,\theta_1(u+v)\}\,(vdv - udu)}{6v(v^2-u^2)} - \frac{\theta_2(u-v) - \theta_1(u+v)}{6v}\,du$$

$$= \frac{\theta_2(u-v)}{6(u-v)}\,d(u-v) + \frac{\theta_1(u+v)}{6(u+v)}\,d(u+v).$$

The integral is

$$z = \phi(u+v) + \psi(u-v)$$

$$= \phi(-3y + \sqrt{9y^2 + 4x^3}) + \psi(-3y - \sqrt{9y^2 + 4x^3}),$$

where ϕ and ψ are arbitrary functions.

Putting $-3y + \sqrt{9y^2 + 4x^3} = 2u_1$, $-3y - \sqrt{9y^2 + 4x^3} = 2v_1$, the general integral becomes

$$z = \phi_1(u_1) + \psi_1(v_1),$$

while the cubic transforms into

$$z^3 - 3z\sqrt[3]{u_1 v_1} - (u_1 + v_1) = 0.$$

The initial values for the cubic are

u_1	v_1	z
0	0	0
0	v_1	$\sqrt[3]{v_1}$
u_1	0	$\sqrt[3]{u_1}$.

Substituting these values in the general integral,

$$0 = \phi_1(0) + \psi_1(0),$$
$$\sqrt[3]{v_1} = \phi_1(0) + \psi_1(v_1),$$
$$\sqrt[3]{u_1} = \phi_1(u_1) + \psi_1(0).$$

By addition

$$\sqrt[3]{u_1} + \sqrt[3]{v_1} = \phi_1(u_1) + \psi_1(v_1);$$

hence

$$z = \sqrt[3]{u_1} + \sqrt[3]{v_1};$$

restoring the original variables,

$$z = \sqrt[3]{\frac{-3y + \sqrt{9y^2 + 4x^3}}{2}} + \sqrt[3]{\frac{-3y - \sqrt{9y^2 + 4x^3}}{2}}.$$

Since $\sqrt[3]{u_1 v_1} = -x$, the admissible roots are

$$z_1 = \sqrt[3]{u_1} + \sqrt[3]{v_1},$$
$$z_2 = w\sqrt[3]{u_1} + w^2\sqrt[3]{v_1},$$
$$z_3 = w^2\sqrt[3]{u_1} + w\sqrt[3]{v_1},$$

where $1 + w + w^2 = 0$.

4. Solution of the Biquadratic. For the biquadratic

$$z = 3nx + 4qy, \qquad p = 3rx + 3p + 4sy.$$

Put $A = 3rx + 4sy + 2p = 0$, $B = 3ax + 4\beta y + 5r = 0$, $C = \alpha + \delta x = 0$. Then

$$Bx - Cx^2 - A = 2ax^2 - \delta x^3 + 4\beta xy + 2rx - 4sy - 2p = 0$$

is the equation that will be considered. Changing the variables from x and y to ξ_1 and ξ_2, the coefficients of $\frac{\delta^3 z}{\delta \xi_1^3}$ and $\frac{\delta^3 z}{\delta \xi_2^3}$ will vanish if

$$\frac{\partial \xi_1}{dy} = m_1 \frac{\partial \xi_1}{\partial x}, \qquad \frac{\partial \xi_2}{dy} = m_2 \frac{\partial \xi_2}{\partial x},$$

where m_1 and m_2 are roots of the cubic equation

$$\lambda^3 - \frac{4y}{x^2}\lambda - \frac{2}{x} = 0.$$

$$m_1 = \frac{2}{x}\left[\sqrt[3]{\frac{x^2}{8} + \sqrt{\frac{x^4}{64} - \frac{y^3}{27}}} + \sqrt[3]{\frac{x^2}{8} - \sqrt{\frac{x^4}{64} - \frac{y^3}{27}}}\right],$$

$$m_2 = \frac{2}{x}\left[w\sqrt[3]{\frac{x^2}{8} + \sqrt{\frac{x^4}{64} - \frac{y^3}{27}}} + w^2\sqrt[3]{\frac{x^2}{8} - \sqrt{\frac{x^4}{64} - \frac{y^3}{27}}}\right].$$

Using the simplest values for ξ_1 and ξ_2, we have

$$\xi_1 = \sqrt[3]{\frac{x^2}{8} + \sqrt{\frac{x^4}{64} - \frac{y^3}{27}}} + \sqrt[3]{\frac{x^2}{8} - \sqrt{\frac{x^4}{64} - \frac{y^3}{27}}},$$

$$\xi_2 = w\sqrt[3]{\frac{x^2}{8} + \sqrt{\frac{x^4}{64} - \frac{y^3}{27}}} + w^2\sqrt[3]{\frac{x^2}{8} - \sqrt{\frac{x^4}{64} - \frac{y^3}{27}}}.$$

The differential equation now becomes

$$\frac{\partial^3 z}{\partial\xi_1^2\partial\xi_2} - \frac{\partial^3 z}{\partial\xi_1\partial\xi_2^2} = 0,$$

and has for its solution

$$z = \phi(\xi_1) + \psi(\xi_2) + \theta(\xi_3),$$

where ξ_1, ξ_2, ξ_3 are roots of the cubic equation

$$\xi^3 - y\xi - \frac{x^2}{4} = 0.$$

The biquadratic may be written

$$z^4 + 8z\epsilon\sqrt{\xi_1\xi_2\xi_3} - 4(\xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1) = 0,$$

where $\epsilon^2 = 1$ and ϵ has the sign of x.

Permuting ξ_1, ξ_2, ξ_3 leaves the biquadratic unchanged, hence

$$z = \phi(\xi_1) + \phi(\xi_2) + \phi(\xi_3).$$

Substituting this value of z in the biquadratic, we have

$$z^4 - 2\left[\phi^2(\xi_1) + \phi^2(\xi_2) + \phi^2(\xi_3)\right] z^2 - 8\phi(\xi_1)\,\phi(\xi_2)\,\phi(\xi_3)\,z$$
$$+ \left[\phi^2(\xi_1) + \phi^2(\xi_2) + \phi^2(\xi_3)\right]^2$$
$$- 4\left[\phi^2(\xi_1)\phi^2(\xi_2) + \phi^2(\xi_2)\phi^2(\xi_3) + \phi^2(\xi_3)\phi^2(\xi_1)\right] = 0.$$

Identifying this equation with the biquadratic,

$$\phi^2(\xi_1) + \phi^2(\xi_2) + \phi^2(\xi_3) = 0,$$

$$\phi(\xi_1)\,\phi(\xi_2)\,\phi(\xi_3) = -\epsilon\sqrt{\xi_1\xi_2\xi_3},$$

$$\phi^2(\xi_1)\,\phi^2(\xi_2) + \phi^2(\xi_2)\,\phi^2(\xi_3) + \phi^2(\xi_3)\,\phi^2(\xi_1) = \xi_1\xi_2 + \xi_2\xi_3 + \xi_3\xi_1,$$

together with $\xi_1 + \xi_2 + \xi_3 = 0$.

Hence

$$\phi(\xi_1) = \epsilon_1\sqrt{\xi_1},\ \phi(\xi_2) = \epsilon_2\sqrt{\xi_2},\ \phi(\xi_3) = \epsilon_3\sqrt{\xi_3},$$

where $\epsilon_1^2 = \epsilon_2^2 = \epsilon_3^2 = 1$, and $\epsilon_1\epsilon_2\epsilon_3 = -\epsilon$.

The solution of the biquadratic is therefore

$$z = -\epsilon_1\sqrt{\xi_1} + \epsilon_2\sqrt{\xi_2} + \epsilon_3\sqrt{\xi_3}.$$

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THE IN- AND-CIRCUMSCRIBED QUADRILATERAL

By W. E. Byerly

In 1792 Fuss* showed that if the radii r and R of two circles and the distance a between their centres satisfy the condition

$$[(R+a)^2 - r^2][(R-a)^2 - r^2] = r^4, \qquad [\text{I}]$$

a quadrilateral can be drawn that will be inscribed in the latter and circumscribed about the former.

Six years later Fuss† carried his investigation further and obtained the conditions that certain polygons of more than four sides could be inscribed in one of two circles and circumscribed about the other.

In 1822 Poncelet in his *Proprietés Projectives* proved a theorem that greatly increased the value of Fuss's results:—"If a polygon of n sides be inscribed in one conic and circumscribed about another an infinite number of polygons of n sides can be inscribed in the first conic and circumscribed about the second, and moreover if from any point on the circumscribed conic a tangent be drawn to the inscribed conic, from the point where this tangent again meets the circumscribed conic a second tangent be drawn, and this process be continued, the nth tangent will intersect the first on the circumscribed conic and a closed polygon of n sides will be formed."

In 1860 Mention‡ took up the problem of the polygon of n sides inscribed in a circle and circumscribed about a second circle and solved it for values of n from 3 to 11, giving as the condition for the case $n = 4$

$$1 - \frac{(R+r+a)(R+r-a)(R-r+a)(R-r-a)}{r^4} = 0. \qquad [\text{II}]$$

The problem of the In- and-Circumscribed Polygon, or as he calls it the Porism of the In- and-Circumscribed Polygon seems to have greatly interested Cayley who wrote several papers§ on the subject taking up the problem for

* *Nova Acta Petrop.* 1792.

† *Nova Acta Petrop.* 1798.

‡ *Bulletin de l'Acad. de St. Petersburg.* 1860.

§ Cayley's *Collected Works*, vols. I, II, and IV, passim.

two conics and dealing with two circles as a special case. In one of these papers* he parallels Mention's work for values of n from 3 to 9.

For the quadrilateral Cayley's condition is

$$\frac{R^2 - a^2}{r^2}\left[\frac{R^4 + r^4 + a^4 - 2a^2r^2 - 2a^2R^2 - 2r^2R^2}{r^4} - 1\right] = 0. \qquad [\text{III}]$$

Fuss's condition [I] and Mention's condition [II] both reduce to

$$R^4 + a^4 - 2a^2r^2 - 2a^2R^2 - 2r^2R^2 = 0 \qquad [\text{IV}]$$

and are identical. Cayley's condition [III] reduces to

$$(R^2 - a^2)\,[R^4 + a^4 - 2a^2r^2 - 2a^2R^2 - 2r^2R^2] = 0 \qquad [\text{V}]$$

and includes the older ones together with the new condition $R^2 - a^2 = 0$.

In the papers to which I have referred, the quadrilateral problem has been treated incidentally and without detail. I wish to consider it independently and from a decidedly elementary point of view.

Let the equations of the inscribed and circumscribed circles be respectively

$$x^2 + y^2 = r^2, \qquad (1)$$

$$(x - a)^2 + y^2 = R^2, \qquad (2)$$

and suppose that a quadrilateral can be inscribed in (2) and circumscribed about (1).

Then by Poncelet's Theorem we can start at any point of (2) and draw tangents to (1), from the points where these tangents meet (2) a second time we can draw tangents to (1) and they will intersect on (2); and we shall have a quadrilateral inscribed in (2) and circumscribed about (1).

If we take as our starting point $(a + R, 0)$ and draw our quadrilateral as indicated above, it is evident from considerations of symmetry that the last vertex formed will lie on the X axis and therefore that it must be one of the two points in which (2) meets the X axis, i. e. $(a - R, 0)$ or $(a + R, 0)$.

Case I. Suppose the last vertex is at $(a - R, 0)$. Then each of the tangents from $(a + R, 0)$ will form with a tangent from $(a - R, 0)$ an angle inscribed in a semi-circle and therefore a right angle.

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* Cayley's *Collected Works*, vol. IV, p. 292, et seq.

In Figures 1 and 2 the right triangles OTC and $OT'B$ are equal. Hence $OC = |a - R|$.

In the right triangle AOC the legs are $a + R$ and $|a - R|$ and the

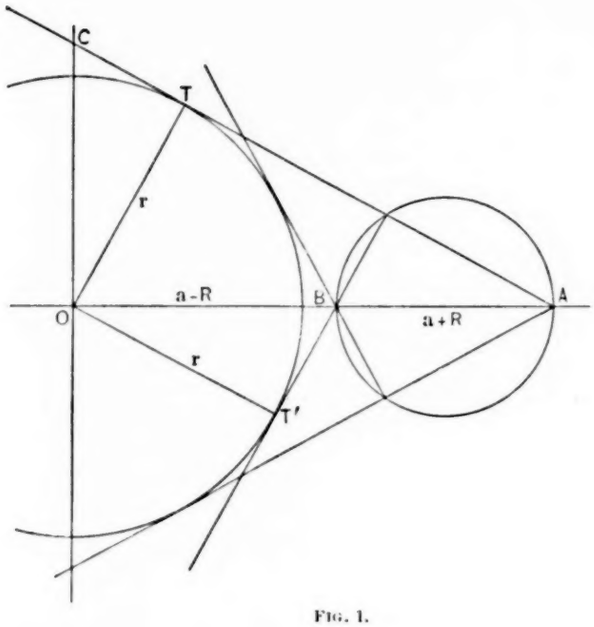

FIG. 1.

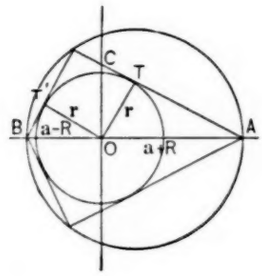

FIG. 2.

distance from the vertex of the right angle to the hypothenuse is r.

Then
$$r = \frac{(a + R)(|a - R|)}{\sqrt{(a + R)^2 + (a - R)^2}},$$

or
$$\frac{1}{r^2} = \frac{1}{(a + R)^2} + \frac{1}{(a - R)^2}. \qquad [VI]$$

This condition [VI], which is equivalent to [I] and [II], the conditions obtained by Fuss and Mention, is therefore a necessary condition. That it is sufficient is easily shown as follows:—In figures 1 and 2 through A draw the tangent AT to the circle (1) and from B drop a perpendicular upon it. Assume [VI]; then $OC = |a - R|$, OTC and $OT'B$ are equal triangles,

$OT' = r$ and therefore BT' is tangent to the first circle. Condition [VI] then is necessary and sufficient.

Case II. Suppose the last vertex is at $(a - R, 0)$.

The second pair of tangents coincide with the first and the quadrilateral is of course a degenerate one.

In this case the points of contact of the tangents from $(a + R, 0)$ must lie on the first circle, perpendiculars to these tangents at their points of contact must pass through $(a - R, 0)$ and also through the centre of the first circle. R must then be equal to a or to $-a$, i. e.,

$$R^2 - a^2 = 0. \qquad \text{[VII]}$$

That [VII] is sufficient as well as necessary may be shown as follows:— Let $R = a$. From A, (a, a), draw a tangent to the first circle and drop a

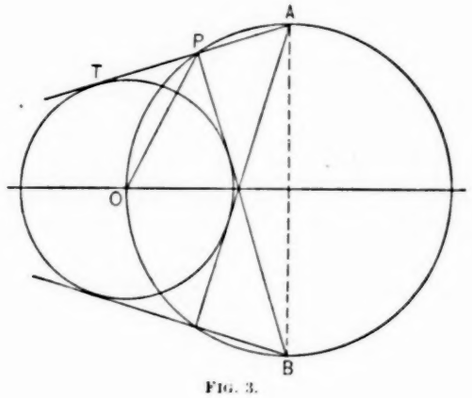

FIG. 3.

perpendicular upon it from B, $(a, -a)$. Join with O the intersection P of these perpendiculars, which of course is a point on the second circle.

The angle OPB, being measured by one half the quadrant OB, is an angle of 45 degrees. OP bisects the angle BPT, and O is equidistant from AT and BP. Therefore BP is a tangent and condition [VII] is sufficient as well as necessary.

The complete condition covering Case I and Case II is, then, [III] the condition given by Cayley.

The following construction by which an inscribed circle corresponding to any given circumscribed circle may be found is interesting.

Join any point P on the circumference of the given circle (Fig. 4) with the extremities A and B of a horizontal diameter and with the extremities C and D of a vertical diameter. Taking as a centre any one of the four points O', O'', C, D, where the second pair of lines, which are easily seen to bisect the angles between the first pair, intersect the horizontal and vertical diameters, with its distance from either of the first pair as a radius, describe a circle and it will be the circle required.

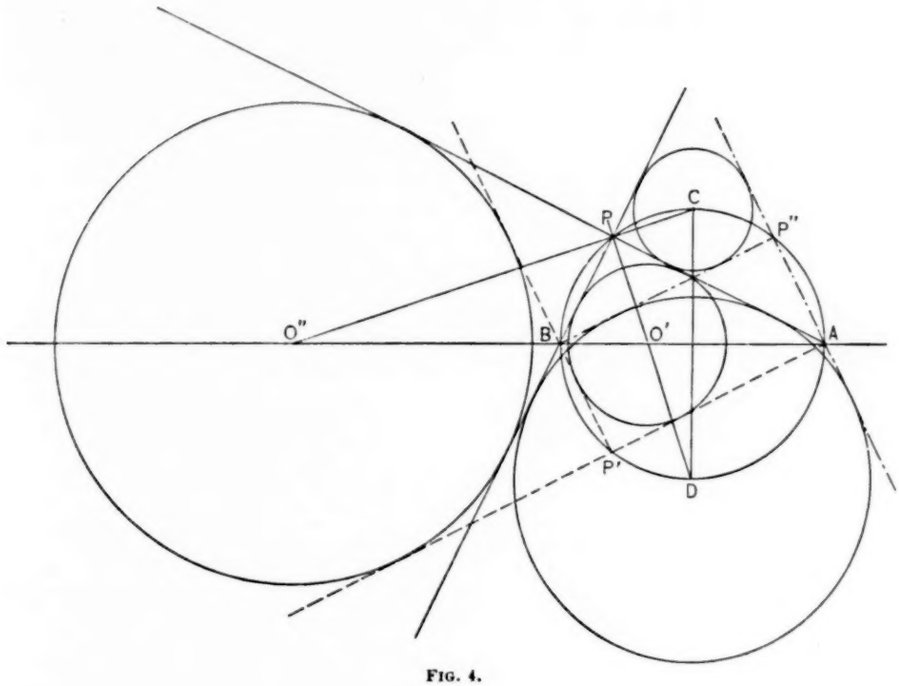

FIG. 4.

The first pair of lines together with their reflections in the horizontal diameter will form a quadrilateral $APBP'$ inscribed in the given circle and circumscribed about the two circles whose centres O' and O'' are in the horizontal diameter.

The first pair of lines together with their reflections in the vertical diameter will form a quadrilateral $APBP''$ inscribed in the given circle and

circumscribed about the two circles whose centres C and D are on the vertical diameter.

The first pair of circles, one of which is wholly within and the other wholly without the given circle, come under Case I and satisfy condition [VI].

The second pair of circles, each of which lies partly within and partly without the given circle, come, under Case [II] and satisfy condition [VII].

Every circle that can be combined with the given circle to form an in- and circumscribed pair can be obtained by the construction just given, followed if necessary by a rotation through a suitable angle about the centre of the given circle.

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APPLICATIONS OF PROBABILITY TO MECHANICS

By Edwin Bidwell Wilson

Introduction. The problem of the application of the theory of probability to the motion of a dynamical system and in particular to finding, under definitely specified assumptions, the *average values* of the important quantities connected with the motion lies quite within the domain of investigations in pure mathematics. In a broad way the theory that thus arises may be termed *statistical mechanics;* from a narrower point of view, however, statistical mechanics may be considered as embracing only so much of the theory as may be built upon the special assumptions which lead to the Maxwell law for the distribution of velocities and to the theory of the equipartition of kinetic energy. Hence it is that the advances in statistical mechanics have been made almost exclusively by mathematical physicists at first interested in the kinetic theory of gases and subsequently concerned with finding a mechanical foundation for thermodynamics, a mechanical formulation of entropy and the law of degradation or dissipation of energy.*

The treatment of these physical analogies requires the systems which are taken into consideration to have a very great number of degrees of freedom and consequently introduces into the subject of statistical mechanics, in its broadest sense, a complexity which is by no means an essential part of it and which goes a long way toward making difficult the acquisition of the elements of the theory. To set forth, as simply as may be, the essential and elementary steps in the application of the theory of probabilities to dynamical systems is the object of these pages; and for this reason the number of degrees of freedom of the systems considered will at first be kept as small as possible. The amount of mathematics which is required is small and is amply covered by the discussions of the application of definite integrals to determining average values of quantities continuously distributed, which may be found in the usual texts on integral calculus.†

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* Especial mention may be made of Maxwell, Boltzmann, Planck, Gibbs, Zermelo, O. Reynolds, and Jeans.

† See Byerly's *Integral Calculus*, chap. 15; Todhunter's *Integral Calculus*, chap. 14; Williamson's *Integral Calculus*, chap. 12.

As it is necessary to have clearly in mind what the term probability means when applied to continuously distributed magnitude, it may be well to recall the definitions with the aid of an example. Consider what meaning shall be attributed to the expression "the average ordinate in a semicircle" (which may be assumed to lie in the first and second quadrants). To find the average of a finite set of quantities $q_1, q_2, \cdots, q_n$ which are repeated respectively $w_1, w_2, \cdots, w_n$ times (the w's are called the weights of the quantities), one writes

$$\bar{q} = \frac{q_1 w_1 + q_2 w_2 + \cdots + q_n w_n}{w_1 + w_2 + \cdots + w_n} = q_1 \frac{w_1}{\Sigma w_i} + q_2 \frac{w_2}{\Sigma w_i} + \cdots + q_n \frac{w_n}{\Sigma w_i},$$

where $\bar{q}$ denotes the average. The second form of q is interpretable in terms of probability. For there are altogether the number Σw_i of the q's and there are w_j of the quantities q_j; and hence the probability or chance that a q picked out at random is q_j is the ratio $w_j/\Sigma w_i$. Thus

$$q = q_1 p_1 + q_2 p_2 + \cdots + q_n p_n$$

is also the value of the average, where p_i denotes the chance or probability of an arbitrary q being q_i.

Now take up the question of the ordinates in the semicircle. There are an infinite number of the ordinates and the probability that an ordinate taken at random be a specified ordinate is zero. The definition of average as given for a finite number of quantities is illusory. To obtain an average it becomes necessary to make some specific assumption as to the distribution of the ordinates. If in the first place we assume that the totality of the ordinates may be represented by a finite number n of them which are uniformly distributed along the diameter so that there is one ordinate y in each equal interval dx of of the diameter or that the number of ordinates is proportional to the length of the portion upon which they stand, then the weight to be attached to the ordinate or ordinates between x and $x + dx$ will naturally be taken to be dx or any multiple kdx of dx and the average will be

$$\bar{y} = \frac{y_1 dx_1 + y_2 dx_2 + \cdots + y_n dx_n}{dx_1 + dx_2 + \cdots + dx_n} = \frac{\Sigma y dx}{\Sigma dx}.$$

In this manner the infinite number of ordinates is really replaced by a finite but very large number which are spaced or distributed in a definite manner. To obtain a definite and unique result the number of ordinates may be imag-

ined to increase without limit and the individual intervals dx to decrease without limit so that the average of the ordinates is

$$\bar{y} = \lim \frac{\Sigma y dx}{\Sigma dx} = \frac{\int_{-a}^{+a} y dx}{\int_{-a}^{+a} dx} = \int_{-a}^{+a} y \frac{dx}{2a} = \frac{\pi}{4} a,$$

where a is the radius of the semicircle.

It would be possible to make an assumption of distribution which differed from the foregoing. We might for instance imagine that the ordinates were suspended at equal intervals from the circular arc. In this case it is clear that there must be relatively fewer ordinates over a portion of the diameter near the center and relatively more over an equal portion near the extremities. In fact the weights which are now to be attached are proportional to ds, the element of arc on the circumference. Under this assumption the average of the ordinates is

$$\bar{y} = \lim \frac{\Sigma y ds}{\Sigma ds} = \frac{\int_{-a}^{+a} y ds}{\int_{-a}^{+a} ds} = \int_{-a}^{+a} y \frac{ds}{\pi a} = \frac{2}{\pi} a.$$

Thus it is seen how vitally the value of the average of a quantity distributed continuously depends on the assumption which is made relative to the distribution.

Another point to mention is that the probability that an ordinate lies between x and $x + dx$ is the weight divided by the sum (the integral) of the weights, that is, it is

$$P dx = \frac{dx}{\int_{-a}^{+a} dx} = \frac{dx}{2a} \quad \text{and} \quad P dx = \frac{ds}{\int_{-a}^{+a} ds} = \frac{ds}{\pi a}$$

in the two cases discussed above. It is of course an infinitesimal with dx. The quotient of this infinitesimal probability by the differential dx is called the *probability function* or the *frequency*; thus under the two above assumptions we have respectively

$$P = \frac{1}{2a} \quad \text{and} \quad P = \frac{1}{\pi a} \frac{ds}{dx} = \frac{1}{\pi \sqrt{a^2 - x^2}}.$$

The only difference between the probability Pdx and the weight is that the weight may contain an arbitrary factor of proportionality whereas the probability is limited by the relation that the sum of all the probabilities represents a certainty, that is, is equal to unity. In this way the value of the factor of proportionality is fixed.

1. Time-averages in a dynamical system with one degree of freedom. Consider the motion of the simple pendulum, or better, the motion of a cycloidal pendulum which may execute vibrations of any magnitude and yet be governed by the equations of simple harmonic motion. Let the mass of the oscillating particle * be taken as unity. Then

$$\frac{d^2x}{dt^2} = -n^2x, \qquad v = \frac{dx}{dt} = n\sqrt{a^2 - x^2}, \qquad x = a\sin(nt + \gamma).$$

What is the probability that the vibrating particle be found between the values x and $x + dx$ of its coordinate? The time that the particle spends in the interval dx is obtained from its velocity as

$$dt = \frac{dx}{v} = \frac{dx}{n\sqrt{a^2 - x^2}}.$$

Hence if the natural assumption be made that the probability of the particle being in the interval dx is proportional to the time spent in traversing that interval, the probability will be

$$\frac{kdx}{n\sqrt{a^2 - x^2}}, \qquad \text{with} \qquad \int_{-a}^{+a} \frac{kdx}{n\sqrt{a^2 - x^2}} = 1$$

as an additional condition due to the fact that the particle must surely lie between the limits $-a$ and $+a$ which give the amplitude of the vibration. The determination of k gives the value

$$k = \frac{1}{\int_{-a}^{+a} \frac{dx}{n\sqrt{a^2 - x^2}}} = \frac{n}{\pi}.$$

* We shall speak of a particle merely for the convenience of having a definite concrete conception before us. It should be distinctly noted, however, that the discussion depends in no way upon this convention, but holds equally well for any system whose motion is adequately represented by the equation $\ddot{x} = -n^2x$.

This determination of the probability fixes the value of the average of quantities connected with the motion. For if u be any quantity of which the average is desired, it is merely necessary to form the integral of u times the probability over the whole interval. Thus

$$\bar{u} = \int_{-a}^{+a} u \frac{n}{\pi} \frac{dx}{n\sqrt{a^2 - x^2}}.$$

If it be desired to average u over only a part of the interval, the probability for that partial interval need not be computed; it suffices to regard the probability already obtained, or any quantity proportional to it, as a weight and write

$$u = \frac{\int_{\alpha}^{\beta} u \frac{dx}{\sqrt{a^2 - x^2}}}{\int_{\alpha}^{\beta} \frac{dx}{\sqrt{a^2 - x^2}}}, \qquad -a \leqq \alpha < \beta \leqq +a.$$

Thus the average value of the square of the displacement is

$$\bar{x^2} = \frac{1}{\pi} \int_{-a}^{+a} \frac{x^2 dx}{\sqrt{a^2 - x^2}} = \tfrac{1}{2} a^2;$$

and the average value of the displacement itself is

$$\bar{x} = \frac{1}{\pi} \int_{-a}^{+a} \frac{x dx}{\sqrt{a^2 - x^2}} = 0,$$

as was to be expected from the physically obvious fact that the particle vibrates in a manner symmetrical with regard to the origin $x = 0$.

Again, if it be desired to find the average value of the displacement taken without regard to sign, it is possible, in view of the symmetry, to take the average of x over positive values and write

$$\overline{|x|} = \frac{\frac{1}{\pi} \int_0^a \frac{x dx}{\sqrt{a^2 - x^2}}}{\frac{1}{\pi} \int_0^a \frac{dx}{\sqrt{a^2 - x^2}}} = \frac{\frac{1}{\pi} a}{\frac{1}{2}} = \frac{2}{\pi} a.$$

The same procedure would be possible in case of any odd power of x. In a similar manner the average velocity of the particle may be found as

$$\bar{v} = \frac{\frac{1}{\pi}\int_0^a \frac{v\,dx}{\sqrt{a^2 - x^2}}}{\frac{1}{2}} = \frac{2n}{\pi}\int_0^a dx = \frac{2}{\pi}na.$$

To find the average of the kinetic energy the whole interval may be considered. Then

$$\tfrac{1}{2}\overline{v^2} = 2\frac{1}{\pi}\int_{-a}^{+a} n^2\sqrt{a^2 - x^2}\,dx = \tfrac{1}{4}\,n^2a^2.$$

Such averages as these which have been obtained are called *time-averages* because they are evaluated upon the supposition that the probability or weights have been determined by the assumption that the weight is proportional to the time spent in traversing the interval.

That the average is obtained on the supposition that the time is the variable according to which the distribution is uniform may also be seen from the formula for the probability. With the time as independent variable the infinitesimal probability would be written as $P\,dt$. Hence it follows that

$$P\,dt = \frac{1}{\pi}\frac{dx}{\sqrt{a^2 - x^2}} = \frac{1}{\pi}\frac{1}{\sqrt{a^2 - x^2}}\frac{dx}{dt}\,dt = \frac{n}{\pi}\,dt,$$

and the probability function or frequency appears as the constant n/π and indicates uniform distribution with respect to t. The process of finding the average of any quantity u may be likened to that of finding the average ordinate in the semicircle. For if u, which is some quantity connected with the motion and hence is a function of t, be plotted as the curve $u = u(t)$ between the limits $t = -\gamma/n$ and $t = \pi/n - \gamma/n$ which correspond to a complete vibration of the particle from $x = -a$ to $x = +a$, then the average ordinate of the curve will be precisely the time-average of u, provided the average ordinate is found on the assumption of uniform distribution of the ordinates relatively to t.

Frequently, however, it is preferable to follow the vibrating particle in space instead of in time and to consider a quantity u connected with the motion as a function $u = u(x)$ of x rather than of t. The infinitesimal probability when x is the variable is $P\,dx$ and

$$P\,dx = \frac{1}{\pi}\frac{dx}{\sqrt{a^2 - x^2}}, \qquad P = \frac{1}{\pi}\frac{1}{\sqrt{a^2 - x^2}}.$$

Under this assumption the probability function is not constant but is inversely proportional to the velocity. From the work done above in determining average values it is clear that the time-averages were in reality obtained with x as the variable although t is the variable with respect to which distribution is uniform. The average of u is not the average ordinate in the curve $u = u(x)$ between the limits $x = -a$ and $x = +a$ when the ordinates are considered as distributed uniformly in x; the density of distribution must be inversely proportional to the velocity which is a function of x.

Thus far it has been a single vibrating particle which has been under consideration and has been followed whether in time or in space over a complete swing or part of a swing for the purpose of obtaining the averages of quantities connected with its motion. For purposes of generalization it is convenient somewhat to change the point of view. Instead of following the particle in space (and thus following it implicitly in time) let us suppose that at any instant there are an infinite number of particles distributed throughout the interval from $x = -a$ to $x = +a$, each one moving exactly as the single particle heretofore considered would move if at the instant considered it happened to be passing through the position of that particle. This amounts to changing from a single particle with a definite phase-angle γ to a set or *ensemble* of particles with different phase-angles but otherwise entirely alike. Furthermore let the density D of the distribution of particles through the interval be proportional to the probability function so that

$$D \propto P = \frac{1}{\pi} \frac{1}{\sqrt{a^2 - x^2}}.^*$$

Now it appears at once that for the average of any quantity u taken for all the particles of the ensemble the weights will be Ddx† and hence that

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* Of course it would, strictly speaking, be impossible to distribute the particles continuously. This, however, is a difficulty which arises in any treatment of a body as a continuous distribution of matter or mass. The density here is a linear density and the number of particles or the amount of matter in the interval dx is Ddx as in the ordinary treatment of a chain or rod.

† For Ddx represents the amount of matter or number of particles in the interval dx. The conception is entirely similar to that previously mentioned in connection with the fact that Pdx was the number of ordinates in the interval dx when P was the frequency or density of the ordinates.

$$\bar{u} = \frac{\int_{-a}^{+a} u\,D dx}{\int_{-a}^{+a} D dx} = \frac{\int_{-a}^{+a} u\,P dx}{\int_{-a}^{+a} P dx} = \int_{-a}^{+a} u\,P dx \qquad \text{(since } D \propto P\text{)}$$

and that in like manner the average taken over a part of the ensemble will be

$$\bar{u} = \frac{\int_{\alpha}^{\beta} u\,D dx}{\int_{\alpha}^{\beta} u\,D dx} = \frac{\int_{\alpha}^{\beta} u\,P dx}{\int_{\alpha}^{\beta} P dx}.$$

From the above formulas it is seen that the average values of u taken over the ensemble or any part of it are identical with the corresponding averages found by following a single particle. This identity is not fortuitous but arises necessarily from the law of density assumed for the distribution of the particles in the ensemble. The advantages of passing from time-averages to *ensemble-averages* will appear more clearly in the next section. One point should be noted here. The density of the distribution of the particles has been assigned at a specified instant. From that instant on, each particle will execute a simple harmonic motion and therefore at any subsequent instant the density of the distribution will be determined by the original distribution and by the motion of the individual particles. It is extremely important, in fact it is essential, that the density of distribution shall not change with the time; otherwise if the ensemble-averages were taken at a later time, the averages would not in general be identical with the time-averages.

Suppose that D is the density of distribution of particles in the ensemble. Consider any interval dx between $x = -a$ and $x = +a$. The number of particles in this interval is $\int_x^{x+dx} D\delta x$,* and it is desired to express the fact or the condition that there is no change in this number owing to the motion of the particles, that is, that as many particles are brought into the interval dx as are removed from it during each interval of time. For convenience let the interval of time δt be chosen as small, small even relatively to the time that it takes a particle to move over the interval dx, and let the velocity of the

* The symbol δx is used in place of dx in the integrand to avoid confusion with the total interval dx.

particles be positive. In the time δt the particles which are in the neighborhood of the point x have the velocity v and will move through the distance $\delta x = v\delta t$. The number of particles in the interval $\delta x = v\delta t$ just to the left of x is $D\delta x = Dv\delta t$ and it is this number which will advance into the interval dx in the time δt . The velocity of particles in the neighborhood of $x + dx$ is $v + dv$ and the distance they will move is $\delta' x = (v + dv)\delta t$. The number of particles in the interval $\delta' x = (v + dv)\delta t$ just to the left of $x + dx$ is $(D + dD)\ \delta' x = (D + dD)\ (v + dv)\delta t$ and it is this number which will advance out of the interval dx in the time δt. Hence if the number in the interval is to remain constant, it is necessary and sufficient that

$$(D + dD)(v + dv)\delta t - Dv\delta t = 0.$$

The condition thus obtained may be simplified by discarding δt, by expanding and canceling Dv, and by omitting the infinitesimal $dDdv$ of the second order. The result is the differential equation

$$vdD + Ddv = 0, \text{ of which } Dv = \text{constant}$$

is the integral. It therefore appears that the necessary and sufficient condition that the distribution of the particles remain unchanged owing to their motion is that the density be inversely proportional to the velocity. As this was the law assumed above, it is clear that that distribution was *permanent* in time. Such a distribution is said to be in *statistical equilibrium*. The result $Dv =$ const. just obtained depends only on the facts that the particle has one degree of freedom and that the velocity is a function of the coordinate specifying the position of the particle.* In such mechanical systems the only possible distribution in statistical equilibrium is that in which the density is inversely proportional to the velocity, and averages taken over the ensemble thus distributed are identical with the time-averages found by following a given particle throughout its motion.

2. Ensemble-averages in a dynamical system with one degree of freedom. In the foregoing section it has been assumed that a definite particle was under consideration and that consequently the amplitude a of the vibration was known. To know the phase-angle was unimportant because the particle was followed throughout a vibration. The ensemble which was introduced was obtained by disregarding the phase-angle and assuming an infinity of

* It should be noted that the conception of statistical equilibrium as set forth in these paragraphs requires the motion to be conservative.

particles all with the same amplitude (and consequently with the same energy) but distributed over all possible phase-angles. This means that one constant of integration, the constant γ, has been regarded as unknown and that a distribution relative to it has been introduced to make up for the lack of knowledge. Suppose next that all that is known about the particle is that it satisfies the differential equation of simple harmonic motion with a given value for n. The initial conditions which determine the constants of integration a and γ are unknown; no particular particle can be followed in its motion. It is necessary to replace the particle by an ensemble in which distribution with regard to both constants of integration is allowed. This ensemble will be a two-dimensional ensemble, not a one-dimensional ensemble as before.

The initial conditions which determine the motion of any particular particle are the position and the velocity at any instant: these may be interpreted as the coordinates of a point in an auxiliary x-v-plane. Thus at the given initial instant there will correspond to each possible motion regulated by the given differential equation a definite point of the x-v-plane, and conversely to each point of the plane (possibly only within certain restricted regions of the plane) there will correspond some particular motion with specific amplitude and phase-angle. The point (x, v) will be called the *representative* point of the motion. As time goes on, the representative point for any particular motion will change from its initial position and will describe a curve in the x-v-plane: for both the position and the velocity of a given particle are functions of the time. In the case of simple harmonic motion these curves will be the ellipses $n^2x^2 + v^2 = n^2a^2$. Nothing as yet has been said as to the density assumed for the distribution of the representative points in the x-v-plane at the initial instant.* That requires careful consideration.

In the first place let it be noted that what the density will ultimately be used for is to determine average values over the ensemble according to the rule

$$u = \frac{\int\int u D dx dv}{\int\int D dx dv},$$

* It should be remarked that here we are speaking of the density of representative points and are not thinking of the density of the particles as in the previous section. In fact it might have been better though less concrete to have spoken in the first instance of representative points and of their density of distribution rather than of the particles themselves.

where u is the quantity to be averaged, where $Ddxdv$ is the product of the density by the element of area and consequently represents the number of motions whose representative points lie in that element, and where the double integrals may be extended over any region of the x-v-plane which contains the points representative of those motions for which the average is desired. If the density D of the distribution at the initial instant were assumed at random, the motion of the representative points would in the course of time bring about a new distribution which in all probability would differ from the original distribution; the averages at the later instant would then probably differ from the original averages. To avoid these inconveniences it is desirable to impose upon the density D the restriction that it shall represent a permanent state, that is, that the ensemble of representative points shall be so distributed as to be in statistical equilibrium.*

The conditions for statistical equilibrium are not hard to find. The most general assumption for the density D would be that D is a function $D(x, v, t)$ of the position, velocity and time. Now if the distribution is permanent, the density cannot depend explicitly on the time, and $\partial D/\partial t = 0$. At a given time the number of representative points within a given rectangle R of the x-v-plane between x and $x + dx$ and v and $v + dv$ is

$$\int_x^{x+dx} \int_v^{v+dv} D(x, v)\,\delta x\,\delta v.$$

The number of points can change only by some of the points moving across one of the four sides of the rectangle. As $\delta x = v\delta t$, the points which will move into R across the left hand side of length dv are those situated in the small rectangle† of area $\delta x dv$ just to the left of R; their number is $Dv\,\delta t\,dv$. The points which will move out across the right hand side of length dv at $x + dx$ must be those in the small rectangle of area $\delta' x dv$ just to the left of that side and, as v is here one of the independent variables and is constant along lines

* Attention may again be called to the fact that the conception of statistical equilibrium requires the motion to be conservative; for if the motion were dying out, it would not have the necessary characteristic of permanency in time.

† The same assumption as before relative to the magnitude of δt will be made, namely that δt is small compared with the time it would take one of the points to move across the rectangle R. Moreover, owing to the smallness of δx as compared with dv or dx, the number of points which might move out or in across the short sides δx of the small rectangle may be neglected.

parallel to the x-axis, $\delta' x = \delta x = v\delta t$; the number of the points is therefore $\left(D + \frac{\partial D}{\partial x}\, dx\right) v\delta t\, dv$. As $\delta v = f\delta t$, where f denotes the acceleration which is given by the differential equation as a function of x, the points which will move into R across the lower side of length dx at v are those in the small rectangle of area $\delta v\, dx$ just below R; their number is $Df\delta t\, dx$. The points which will move out across the upper side of length dx at $v + dv$ are those in the small rectangle of area $\delta' v\, dx$ just below that side and, as f is a function of x alone and is constant along lines parallel to the v-axis, $\delta' v = \delta v = f\delta t$; the number of the points is therefore $\left(D + \frac{\partial D}{\partial v}\, dv\right) f\delta t\, dx$. Hence, if the total number of points is to remain unchanged,

$$Dv\delta t\, dv - \left(D + \frac{\partial D}{\partial x}\, dx\right) v\delta t\, dv + Df\delta t\, dx - \left(D + \frac{\partial D}{\partial v}\, dv\right) f\delta t\, dx = 0$$

or

$$\frac{\partial D}{\partial x}\, v + \frac{\partial D}{\partial v}\, f = 0.$$

This is the condition for statistical equilibrium which was desired. It is therefore seen that D must satisfy a definite partial differential equation in x and v.

The solution of the partial differential equation just found depends upon the solution of the simultaneous set of equations*

$$\frac{dx}{v} = \frac{dv}{f} = \frac{dD}{0}.$$

The solution of the first two equations may be obtained at once:

$$f\, dx = v\, dv \qquad \text{or} \qquad mf\, dx = mv\, dv \qquad \text{or} \qquad F\, dx = mv\, dv$$

if the mass of the moving particles is m and F is the force acting on it. The equation may be simplified by writing $F = -\, dV/dx$, where V is the potential energy (the motion is assumed to be conservative). Then

$$-\, dV = d(\tfrac{1}{2}\, mv^2), \qquad \text{and finally} \qquad \tfrac{1}{2}\, mv^2 + V = \text{const.}$$

* It may be noted that if these equations be written in parametric form as

$$\frac{dx}{v} = \frac{dv}{f} = \frac{dt}{1}$$

they define the path of the representative point in the x-v-plane.

Hence the value of D is $D = \Omega\ (\text{const.}) = \Omega\ (E)$, where E is the total energy and Ω is any arbitrary function. It therefore is seen that the condition for statistical equilibrium in the distribution of representative points for possible motions of a conservative dynamical system of one degree of freedom is that the assumed density should be a function of the total energy.

The ensemble-averages will naturally depend upon the choice of the arbitrary function Ω. Suppose that (in the special example under consideration where $f = -n^2x$ and $m = 1$) this is taken as

$$D = \Omega(E) = ke^{-c^2E} = ke^{-c^2(\frac{1}{2}v^2 + \frac{1}{2}n^2x^2)},$$

where k and c are constants; and let it be assumed that any values are permissible for the velocity and position of the particle so that the representative points are spread all over the x-v-plane. The mean square of the displacement x will then be

$$\overline{x^2} = \frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^2 e^{-c^2(\frac{1}{2}v^2 + \frac{1}{2}n^2x^2)}\,dx\,dv}{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-c^2(\frac{1}{2}v^2 + \frac{1}{2}n^2x^2)}\,dx\,dv}.$$

The form of the integrals is such that the integrations with respect to x and v are quite independent. Hence, discarding the integration with respect to v which introduces merely a factor in numerator and denominator, the result may be written as

$$\overline{x^2} = \frac{\int_{-\infty}^{+\infty} x^2 e^{-\frac{c^2}{2}n^2x^2}\,dx}{\int_{-\infty}^{+\infty} e^{-\frac{c^2}{2}n^2x^2}\,dx} = \frac{\frac{1}{c^3n^3}\sqrt{2\pi}}{\frac{1}{cn}\sqrt{2\pi}} = \frac{1^*}{c^2n^2}.$$

The average value of the potential energy would be $1/2c^2$. It may readily be shown that the average of the kinetic energy would also be $1/2c^2$. The average displacement would of course be zero. To find the average of the magnitude $|x|$ of the displacement it would merely be necessary to evaluate

*By the ordinary methods of evaluating these integrals or by B. O. Pierce's *Short Table of Integrals*, formulas 492 and 494 of the revised edition.

$$\overline{|x|} = \frac{\int_0^\infty \int_0 x e^{-c^2(\frac{1}{2}v^2 + \frac{1}{2}n^2x^2)}\, dxdv}{\int_0^\infty \int_0^x e^{-c^2(\frac{1}{2}v^2 + \frac{1}{2}n^2x^2)}\, dxdv} = \frac{1}{cn}\sqrt{\frac{2}{\pi}}.$$

Other averages with this assumption for the density of the distribution and for the extent of the distribution of the representative points in the plane may be obtained if desired.

To connect the present discussion where the energy is allowed to have different values with the preceding case in which the energy was fixed, it is merely necessary to consider those motions whose representative points lie between the curves $E = C$ and $E = C + dC$ of equal energy. In the above example these are similar and similarly situated ellipses. If x and E instead of x and v are taken as the independent variables, the density of distribution takes the form

$$D dx dE = \Omega(E)\, \frac{\partial v}{\partial E}\, dx dE = \frac{\Omega(E)}{m}\, dE\, \frac{dx}{v},$$

where m is the mass of the particle; and hence it appears that if the energy be held constant the density regarded as a function of x is inversely proportional to the velocity. This is in accord with the results of the last section, as was to have been expected. For different energies the density is no longer proportional to the velocity because the factor $\Omega(E)$ is different for different values of E.

Once the density is fixed as $\Omega(E)$, the probability function which is to be taken proportional to the density becomes $P = k\,\Omega(E)$. If the limits of the region throughout which the representative points are assumed to be spread are also assigned, the value of k may be determined by the relation

$$\int\int k\Omega(E) dx dv = 1 \quad \text{or} \quad \int\int \frac{k}{m}\, \frac{\Omega(E)}{v}\, dx dE = 1,$$

where the double integral is extended over the entire region occupied by the representative points: for the total probability is 1. It is thus seen that whereas in the simplest case where there is a constant energy it is most natural to start with the conception of the probability and the time-average and then to proceed to the ensemble and to its density of distribution, in the case of variable energy it is simpler and almost necessary to start with the ensemble in

statistical equilibrium and proceed to the probability. And whereas the time-averages and ensemble-averages were identical in the first case, they are distinct in the second—or better, one might say that the conception of a time-average implied the following of a definite motion or at most a group of motions which differed only by an additive constant for the time and that therefore the conception became illusory in the second case and we were forced to use ensemble-averages.

3. Systems with two degrees of freedom. Further to investigate the advantage or necessity of having recourse to the ensemble in statistical equilibrium consider what happens when averages are desired for a particle or system with two coordinates. To exhibit some of the points at issue it will be sufficient to consider the simple case of harmonic motion relative to two perpendicular axes and defined by the equations

$$\frac{d^2x}{dt^2} = -n^2x, \qquad \frac{d^2y}{dt^2} = -m^2y, \qquad \frac{dx}{dt} = n\sqrt{a^2 - x^2}, \qquad \frac{dy}{dt} = m\sqrt{b^2 - y^2},$$

$$x = a\cos(nt + \gamma), \quad y = b\sin(mt + \delta), \quad T = \tfrac{1}{2}(n^2a^2 + m^2b^2 - n^2x^2 - m^2y^2).$$

As the mass of the particle has been taken as unity, we have $v = \sqrt{2T}$ and the time required for the description of the arc ds is

$$dt = \frac{ds = \sqrt{dx^2 + dy^2}}{\sqrt{n^2a^2 + m^2b^2 - n^2x^2 - m^2y^2}}.$$

If as in the first section the probability be considered as proportional to this time, there results

$$Pdt = \frac{kds}{\sqrt{n^2a^2 + m^2b^2 - n^2x^2 - m^2y^2}}, \qquad \int \frac{kds}{\sqrt{n^2a^2 + m^2b^2 - n^2x^2 - m^2y^2}} = 1,$$

where the integral extends over the entire length of the curve described by the particles.

Now if the constants n and m are commensurable, the particle will describe a closed path, the integral which must be evaluated to determine k will converge, and the value of k may be found. Then the time-averages of any quantities connected with the motion could be found as in the first section. But if m and n are incommensurable, the particle will describe an endless Lissajous' curve, the integral will diverge, the determination of k will

be impossible, and the whole problem will become illusory. It appears, however, that the particle after a sufficient lapse of time may be found as near as desired to its original position and with its velocities v_x and v_y as near as desired to their original values. By disregarding these small differences the path may artificially be considered as closed, a value of k may be computed on this assumption, and the averages may be found. This method of dealing with the problem is possible and might readily be adopted from a mathematical point of view and would undoubtedly give satisfactory physical results. But the method which actually is adopted is to pass over to the consideration of ensembles distributed in statistical equilibrium. This will now be explained.

Four initial conditions are required to specify the motion of the particle, namely, the initial values of x, y, v_x, v_y. Let these four quantities be interpreted as rectangular coordinates in a space of four dimensions, thus obtaining in that space a representative point for the motion. Each point of the space will correspond to some motion* and every motion will have a definite representative point at the initial instant. As time passes, the representative points will describe paths of one dimension, that is, curves in the representative space of four dimensions. Everything is quite similar to the simpler case where there was only one coordinate and the representative space had two dimensions. Here the most general assumption for the density would be that it was a function $D(x, y, v_x, v_y, t)$ of the five variables which enter into the description of the motion. If there is to be statistical equilibrium, that is, if the density is not to depend on the time, considerations like those previously advanced show that D cannot depend explicitly upon t and must satisfy the partial differential equation

$$\frac{\partial D}{\partial x} v_x + \frac{\partial D}{\partial y} v_y + \frac{\partial D}{\partial v_x} f_x + \frac{\partial D}{\partial v_y} f_y = 0.$$

The details of the deduction may be omitted.

The solution of this partial differential equation depends on the solution of the system of ordinary equations

$$\frac{dx}{v_x} = \frac{dy}{v_y} = \frac{dv_x}{f_x} = \frac{dv_y}{f_y} = \frac{dD}{0}.$$

* There may be need of restricting the range of representative points to some region of the four-dimensional space; for the energy must be real and moreover it may be desirable to restrict our discussion to only some of all the possible motions.

There will be three independent integrals of these equations and the density D will be a function, an arbitrary function, of the three. In the particular case considered above the equations become

$$\frac{dx}{v_x} = \frac{dy}{v_y} = -\frac{dv_x}{n^2x} = -\frac{dv_y}{m^2y} = \frac{dD}{0},$$

and the integrals are readily seen to be

$$n^2x^2 + v_x^2 = 2c_1, \qquad m^2y^2 + v_y^2 = 2c_2,$$

$$\frac{1}{n}\cos^{-1}\frac{nx}{\sqrt{n^2x^2 + v_x^2}} - \frac{1}{m}\cos^{-1}\frac{my}{\sqrt{m^2y^2 + v_y^2}} = c_3;$$

and hence if the distribution is to be in statistical equilibrium it is necessary and sufficient that the density be

$$D = \Omega(c_1, c_2, c_3).$$

It will be noticed that the sum of c_1 and c_2 is the total energy E and the density may therefore also be taken as

$$D = \Omega(E, c_1 - c_2, c_3).$$

Averages over the ensemble are obtained by quadruple integration

$$\bar{u} = \frac{\int\int\int\int u\,D\,dx\,dy\,dv_x\,dv_y}{\int\int\int\int D\,dx\,dy\,dv_x\,dv_y}$$

extended over such four-dimensional regions as contain the representative points of the motions for which the average is desired.

It should be noted that the three integrals c_1, c_2, c_3, are invariants for the motion of any given particle, that is, are constant in time when any particular motion is followed. For they satisfy the above set of simultaneous ordinary equations and hence also the partial differential equation which D satisfies. That is,

$$\frac{\partial c_1}{\partial x}v_x + \frac{\partial c_1}{\partial y}v_y + \frac{\partial c_1}{\partial v_x}f_x + \frac{\partial c_1}{\partial v_y}f_y = 0,$$

and so for the other two. But as c_1, and also the others, do not contain the time explicitly, it follows that $\partial c_1/\partial t = 0$ and hence

$$\frac{dc_1}{dt} = \frac{\partial c_1}{\partial t} + \frac{\partial c_1}{\partial x} v_x + \frac{\partial c_1}{\partial y} v_y + \frac{\partial c_1}{\partial v_x} f_x + \frac{\partial c_1}{\partial v_y} f_y = 0,$$

which shows that the total derivative of c_1 with respect to the time is 0. It is likewise evident that any function of these three invariants is an invariant of the motion. Moreover there could not be a fourth invariant which was functionally independent of these three. For if there were four invariants c_1, c_2, c_3, c_4 which were functionally independent, they could be solved for the four arguments x, y, v_x, v_y; they would therefore entirely fix the representative point, and there would be no possibility of passing from one such point to another in following the motion. It is clear that apart from the particular case of harmonic motion there will be for the motion of any system with two degrees of freedom three and only three functionally independent invariants. If there is statistical equilibrium, the density must be a function of these three quantities and as the energy is surely an invariant of the motion, it may be taken as one of the three.

An interesting connection with the theory of time-averages may now be made. For, suppose that the density is chosen as a function of the energy alone, as $D = \Omega(E)$, and let the energy be taken as one of the independent variables replacing v_x. Then the density may be written in the form (if it be assumed that the mass is unity)

$$D dx dy dv_x dv_y dE = \Omega(E) \frac{dx}{v_x} dy dv_y dE = \Omega(E) dy dv_y dE \frac{ds}{v}.^*$$

Now consider all the motions whose representative points lie between the two surfaces (of three dimensions) $E = C$ and $E = C + dC$ in the representative space of four dimensions. For these motions $\Omega(E)$ has the sensibly constant value $\Omega(C)$ and hence it appears that the density is inversely proportional to the velocity with a constant factor of proportionality $\Omega(C)$. As the probability is taken as directly proportional to the density, it is seen that the probability is proportional to the time of describing the arc ds. Hence if averages be obtained by integration over the region between the surfaces $E = C$ and $E = C + dC$, those averages may be considered as time-averages, and are generally so considered although it is clear that motions whose representative points lie between these surfaces are by no means so restric-

* For $\frac{\partial v_x}{\partial E} = \frac{1}{v_x}$ and $dx : \sqrt{dx^2 + dy^2} = v_x : \sqrt{v_x^2 + v_y^2}$.

ted as to differ merely by an additive constant for the time. Thus it appears that the introduction of statistical equilibrium and ensemble-averages has led naturally to a very considerable entension in the idea of a time-average, an extension which at first sight might have seemed to be contradictory to any notion of time-average.

Conclusion. The foregoing discussion has been presented as an introduction to the general theory of statistical mechanics. It is not intended to present that theory here. Readers may refer to such portions of Gibbs's treatise on the subject* as may interest them. It may, however, be stated that the subject is concerned not so much with the physical system which is executing the motion as with the differential equations of the motion. The word particle has been employed so largely merely for definiteness. The work of the text which was connected with the equation $d^2x/dt^2 = -n^2x$ would apply equally well to the vibrations of a torsion pendulum where x represented angular rotation and v represented angular velocity. The problem of article 3 could be interpreted as being that of the vibrations of any system depending on two generalised normal coordinates. Under this class of problems would come the case of a stretched massless string carrying two particles constrained to move in a plane and in lines perpendicular to the position of equilibrium of the string.†

In considering the motion of any conservative system of which the position is specified by n generalised coordinates, it is customary to introduce the generalised momenta in place of the generalised velocities. If $q_1, q_2, \cdots, q_n$ are the generalised coordinates, T the kinetic energy, and V the potential energy, the generalised momenta are

$$p_1 = \frac{\partial T}{\partial \dot{p}_1}, \quad p_2 = \frac{\partial T}{\partial \dot{q}_2}, \cdots, p_n = \frac{\partial T}{\partial \dot{q}_n}$$

by definition. The equations of motion are then

$$\dot{p}_1 = \frac{\partial (T-V)}{\partial q_1}, \cdots, \dot{p}_n = \frac{\partial (T-V)}{\partial q_n}$$

* *Elementary Principles in Statistical Mechanics* by J. Willard Gibbs, 1901.

† This problem is frequently treated as a preparation for considering the string as loaded with n particles where n is allowed to become infinite and thus lead to the vibrations of a violin string according to the method used by Lagrange.

in the Lagrangian form, or in the Hamiltonian form they are

$$\dot{p}_i = -\frac{\partial(T+V)}{\partial q_i}, \quad \dot{q}_i = \frac{\partial(T+V)}{\partial p_i}, \quad i = 1, 2, \cdots, n.$$

There are $2n$ initial conditions corresponding to the initial values of the $2n$ coordinates q and momenta p. These $2n$ quantities may be regarded as the coordinates of a representative point in $2n$ dimensions. The condition for statistical equilibrium is then that the density D of the distribution of these representative points over the auxiliary space of $2n$ dimensions shall satisfy the partial differential equation

$$\frac{\partial D}{\partial q_1}\dot{q}_1 + \frac{\partial D}{\partial p_1}\dot{p}_1 + \cdots + \frac{\partial D}{\partial q_n}\dot{q}_n + \frac{\partial D}{\partial p_n}\dot{p}_n = 0.$$

The solution of this equation for D depends on the solution of the set of simultaneous equations

$$\frac{dq_1}{\dot{q}_1} = \frac{dp_1}{\dot{p}_1} = \cdots = \frac{dq_n}{\dot{q}_n} = \frac{dp_n}{\dot{p}_n} = \frac{dD}{0}.$$

The total energy $E = T + V$ is one integral of these equations and there are $2n - 2$ other integrals $C_1, C_2, \cdots, C_{2n-2}$. The density D is therefore any arbitrary function Ω of these quantities,

$$D = \Omega(E, C_1, C_2, \cdots, C_{2n-2}).$$

It is generally customary to consider the density as a function of the energy alone. The particular function which Gibbs uses and which, under the assumptions which he makes, establishes the principle of the equipartition of kinetic energy, is

$$D \propto P = e^{\frac{E-\psi}{\Theta}}, \text{ where } \Theta \text{ and } \psi \text{ are constants,}$$

and the distribution is then called *canonical*. In the particular case where the attention is concentrated on the motions whose representative points lie between two adjacent surfaces $E = C$ and $E = C + dC$ of constant energy, the distribution is called microcanonical. The advantage of discussing the general canonical case first and the microcanonical case second has been sufficiently illustrated by the treatment of the problems in the text.

Massachusetts Institute of Technology,
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THERMODYNAMIC ANALOGIES FOR A SIMPLE DYNAMICAL SYSTEM

By Edwin Bidwell Wilson

Introduction. In a recent article * I have set forth some of the most elementary notions in regard to the statistical mechanics of dynamical systems. The object of the present communication, which may be regarded as a sequel to the other, is to investigate a simple dynamical system for the purpose of exhibiting certain analogies between its statistical properties and the elementary thermodynamics of gases. The dynamical system which I shall choose is the somewhat familiar household article known as the Japanese or Chinese dinner-gong and consisting of a series of gongs of varying sizes suspended one above another upon a vertical string. The number of degrees of freedom of a set of several gongs is considerable; and to simplify the work it will be assumed that the only motion which the gongs may have is one of rotation about the vertical string as axis. Under this hypothesis the number of degrees of freedom will be equal to the number of gongs in the set.

To find the equations of motion of the system in the Lagrangian or Hamiltonian form, it is first necessary to write down the kinetic and potential energies of the system. If $I_1, I_2, I_3, \cdots$ are the moments of inertia of the successive gongs beginning with the top one and if $q_1, q_2, q_3, \cdots$ are the angles through which the respective gongs have turned from the position of equilibrium in which there is no twist in any of the strings, the angular velocities of the gongs will be respectively $\dot{q}_1, \dot{q}_2, \dot{q}_3, \cdots$ and the kinetic energy of the system will be

$$T = \tfrac{1}{2}(I_1\dot{q}_1^2 + I_2\dot{q}_2^2 + I_3\dot{q}_3^2 + \cdots).$$

The momenta corresponding to the different coordinates are

$$p_i = \frac{\partial T}{\partial \dot{q}_i} = I_i\dot{q}_i \qquad \text{or} \qquad \dot{q}_i = \frac{p_i}{I_i};$$

* Applications of Probability to Mechanics, Annals of Mathematics, volume 10, pp. 129-148 (1909). References to this article will be given simply by the page numbers in parentheses without the quotation of the title.

and hence as a function of the momenta the kinetic energy is

$$T = \epsilon_p = \tfrac{1}{2}\left(\frac{p_1^2}{I_1} + \frac{p_2^2}{I_2} + \frac{p_3^2}{I_3} + \cdots\right).^*$$

For the purpose of getting rid of the constant π in some of the subsequent averages, the assumption will be made that the product

$$I_1 I_2 I_3 \cdots = \frac{1}{2\pi},$$

and the kinetic energy will be taken in the form

$$T = \epsilon_p = \pi(a_1 p_1^2 + a_2 p_2^2 + a_3 p_3^2 + \cdots), \quad a_1 a_2 a_3 \cdots = 1. \qquad (1)$$

The potential energy of the system will consist of the energies stored in the twisted strings. The twists in the successive strings beginning with the top one and running down are

$$q_1,\ q_2 - q_1,\ q_3 - q_2,\ \cdots.$$

The potential energy will be proportional to the square of the twist, where the factor of proportionality may depend on a variety of things such as the kind of string, the diameter of the string, the length of the string, and the tension in the string due to the weight of the gongs. In order that there may be some possible variation of the system as a whole it will be assumed that the length of the uppermost string is adjustable; this could easily be arranged by running the string between two closely pressed wheels, which could be moved up and down along the string, instead of merely pinching it in a vice. It is then important to know at least how the factor of proportionality varies with the length of a uniform string when nothing but the length changes. Now from the symmetry of the figure it appears that to twist a string through an angle q is equivalent to twisting each half of it through the angle $\frac{1}{2}q$ in opposite directions. Hence if the factor of proportionality is supposed to vary with the nth power of the length and is written as kl^n, the equation of the whole potential energy to the energy of the two halves gives

$$kl^n q^2 = 2k\left(\frac{l}{2}\right)^n \left(\frac{q}{2}\right)^2 \qquad \text{or} \qquad 1 = \tfrac{1}{2}\left(\frac{1}{2}\right)^n$$

* The use of ϵ_p for kinetic energy, ϵ^q for potential energy, and $\epsilon = T + V = \epsilon_p + \epsilon_q$ for total energy is adopted to conform to the usages of Gibbs's *Elementary Principles in Statistical Mechanics*.

and it is seen that $n = -1$. The potential energy of the system will therefore be written in the form

$$V = \epsilon_q = k_1 l_1^{-1} q_1^2 + k_2 l_2^{-1}(q_2 - q_1)^2 + k_3 l_3^{-1}(q_3 - q_2)^2 + \cdots.$$

For the sake of simplification let it be assumed that an adjustment between the sort of string and the diameter of the strings and the units of measurement has been made so that

$$k_2 l_2^{-1} = \pi, \quad k_3 l_3^{-1} = \pi, \cdots; \quad \text{and let} \quad k_1 l_1^{-1} = \pi v^{-4},$$

so that the parameter v is proportional to the fourth root of the length of the uppermost string of variable length. Then

$$V = \epsilon_q = \pi[v^{-4} q_1^2 + (q_2 - q_1)^2 + (q_3 - q_2)^2 + \cdots]$$

is the expression for the potential energy of the system.

The differential equations for the motion would be

$$\dot{p}_1 + (v^{-4} + 1)q_1 - q_2 = 0, \quad \dot{p}_2 + 2q_2 - q_1 - q_3 = 0, \cdots$$

or

$$\dot{p}_1 + (v^{-4} + 1)q_1 - q_2 = 0, \quad \dot{q}_1 = a_1 p_1, \quad \dot{p}_2 + 2q_2 - q_1 - q_3 = 0, \quad \dot{q}_2 = a_2 p_2, \cdots$$

by direct substitution in the formulas of the Lagrangian or Hamiltonian equations (see page 147), where

$$T - V = \epsilon_p - \epsilon_q = \pi[a_1 p_1^2 + a_2 p_2^2 + a_3 p_3^2 + \cdots - v^{-4} q_1^2 - (q_2 - q_1)^2 - (q_3 - q_2)^2 \cdots]$$

or

$$T + V = \epsilon_p + \epsilon_q = \pi[a_1 p_1^2 + a_2 p_2^2 + a_3 p_3^2 + \cdots + v^{-4} q_1^2 + (q_2 - q_1)^2, + (q_3 - q_2)^2 + \cdots].$$

If D denotes differentiation with respect to the time, the Hamiltonian equations after the elimination of the momenta are

$$\left[\frac{1}{a_1} D^2 + (v^{-4} + 1)\right] q_1 - q_2 = 0,$$

$$- q_1 + \left[\frac{1}{a_2} D^2 + 2\right] q_2 - q_3 = 0,$$

. . . . . .

Hence, if for example the number of gongs is three, the solution of the differential equations* depends on the solution of

$$\begin{vmatrix} \frac{1}{a_1}D^2 + v^{-4} + 1 & -1 & 0 \\ -1 & \frac{1}{a_2}D^2 + 2 & -1 \\ 0 & -1 & \frac{1}{a_3}D^2 + 1 \end{vmatrix} = 0,$$

which is an algebraic equation of the third degree in D^2. If there had been five gongs the solution would have depended on the solution of an algebraic equation of the fifth degree in D^2. As these equations are necessarily literal when the length of the uppermost string is adjustable and v cannot be assigned a numerical value, the solution of the algebraic equations is impossible in practice. This will make no difference to us inasmuch as the discussion of statistical mechanics is based on the differential equations and not upon their integrals.

1. The canonical distribution. Suppose for the moment that the system consists of three gongs. The state of the system is at any instant completely determined by the values of the six quantities $q_1, q_2, q_3, p_1, p_2, p_3$. The representative point of the motion would therefore be located in a space of six dimensions. Consider an ensemble of these dinner-gong systems distributed with the density or better with the probability (see page 148)

$$P = e^{\frac{\psi - \epsilon}{\Theta}}, \quad \epsilon = T + V = \epsilon_p + \epsilon_q, \quad \psi \text{ and } \Theta \text{ constant},$$

where

$$\epsilon = \pi\,[a_1 p_1^2 + a_2 p_2^2 + a_3 p_3^2 + v^{-4} q_1^2 + (q_2 - q_1)^2 + (q_3 - q_2)^2].$$

The distribution will be in statistical equilibrium because the probability, or density which is proportional to it, is a function of the energy alone and the energy is an invariant of the motion. Let it further be assumed that the dis-

* See Forsyth's *Differential Equations*, 3rd edition, p. 305. Or Cohen's *Differential Equations*, p. 150. Or any treatment of simultaneous linear differential equations with constant coefficients.

tribution covers the whole of the six-dimensional representative space so that the relation

$$\int\int\int_{-\infty}^{+\infty}\int\int\int e^{\frac{\psi-\epsilon}{\Theta}}\,dp_1dp_2dp_3dq_1dq_2dq_3 = 1 \tag{2}$$

must be satisfied (see pages 132, 142). This distribution will be considered as the *canonical* distribution and averages over it will be investigated.*

The equation (2) may be used to determine the constant ψ in terms of the other constants of the problem. For as ψ and Θ are constants, the factor $e^{\frac{\psi}{\Theta}}$ may be passed out from under the sign of integration and may be transferred to the other side of the equation. Then

$$e^{-\frac{\psi}{\Theta}} = \int\int\int_{-\infty}^{+\infty}\int\int\int e^{-\frac{\epsilon}{\Theta}}\,dp_1dp_2dp_3dq_1dq_2dq_3 =$$

$$\int_{-\infty}^{+\infty} e^{-\frac{\pi p_1^2}{a_1\Theta}}dp_1\int_{-\infty}^{+\infty} e^{-\frac{\pi p_2^2}{a_2\Theta}}dp_2\int_{-\infty}^{+\infty} e^{-\frac{\pi p_3^2}{a_3\Theta}}dp_3\ \times$$

$$\int_{-\infty}^{+\infty} e^{-\frac{\pi q_1^2}{r^4\Theta}}dq_1\int_{-\infty}^{+\infty} e^{-\frac{\pi(q_2-q_1)^2}{\Theta}}dq_2\int_{-\infty}^{+\infty} e^{-\frac{\pi(q_3-q_2)^2}{\Theta}}dq_3.$$

Here the six successive integrations are to be carried out by beginning with the one for q_3 and working backward through the list. As q_2 is regarded as

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* As a general comment on the assumptions underlying the foregoing use of a canonical distribution, one point in particular should be mentioned. It has been assumed that the representative points for the motion of the systems in the ensemble are spread over the whole of the representative space, that is, that the coordinates $q_1, q_2, q_3, \ldots$ and momenta $p_1, p_2, p_3, \ldots$ may have any value. This necessitates the admission that the angular velocities of the discs and the twists in the strings may be indefinitely great. It goes without saying that this is physically impossible: for if the strings were allowed to twist very much, the law of torsion would depart from the simple linear (Hooke's) law and the expression of the potential energy would change; furthermore the twisting would perhaps materially shorten the strings and introduce complications on that score. It should be noted, however, that as the coordinates and momenta increase, the energy increases with their square, and the probability assumed for the distribution decreases exponentially with the energy. Hence the probability of systems with large velocities and twists is almost negligibly small compared with that of systems with small velocities and twists. The errors made by entirely neglecting the systems in which the velocities or twists were so large as to introduce physical difficulties would therefore probably be small and could certainly be made small by restricting Θ to sufficiently small values. Hence there is no need of worrying very much over such difficulties.

constant for the integration with respect to q_3, we may write

$$\int_{-\infty}^{+\infty} e^{-\frac{\pi(q_3-q_2)^2}{\Theta}} dq_3 = \int_{-\infty}^{+\infty} e^{-\frac{\pi(q_3-q_2)^2}{\Theta}} d(q_3-q_2) = \frac{\sqrt{\Theta}}{\sqrt{\pi}} \sqrt{\pi} = \sqrt{\Theta}.$$

The result is independent of q_2; in fact the integrals may be regarded as multiplied instead of iterated. The result is

$$e^{-\frac{\psi}{\Theta}} = \sqrt{a_1 a_2 a_3}\ v^2 (\sqrt{\Theta})^6 = v^2 \Theta^3,$$

the last reduction being due to the assumption (1). Hence

$$\psi = -2\Theta \log v - 3\Theta \log \Theta. \tag{3}$$

It is readily seen that if there had been n gongs, the expression for ψ would have been

$$\psi = -2\Theta \log v - n\Theta \log \Theta. \tag{3'}$$

Another important quantity is the index η of probability which is defined as the logarithm of the probability. Thus

$$\eta = \log P = \frac{\psi - \epsilon}{\Theta}, \qquad P = \epsilon^{\eta}.$$

The average value of the index of probability η taken over the ensemble is found by evaluating the integral

$$\bar{\eta} = \int_{-\infty}^{+\infty} \cdots \int \eta e^{\eta} dp_1 \cdots dq_3.$$

This may be evaluated directly; but the work may be very much shortened by adopting the device of differentiating (2) with respect to Θ regarded as a parameter. Upon the left-hand side, the differentiation may by Leibniz's rule be carried under the sign of integration. The result is

$$\int_{-\infty}^{+\infty} \cdots \int \frac{\partial}{\partial \Theta} e^{\frac{\psi-\epsilon}{\Theta}} dp_1 \cdots dq_3 = \int_{-\infty}^{+\infty} \cdots \int \left[-\frac{\psi-\epsilon}{\Theta^2} + \frac{1}{\Theta}\frac{\partial \psi}{\partial \Theta} \right] e^{\frac{\psi-\epsilon}{\Theta}} dp_1 \cdots dq_3 = 0.$$

Now Θ may be canceled out and the result transposed so that

$$\frac{\partial \psi}{\partial \Theta} \int_{-\infty}^{+\infty} \cdots \int e^{\frac{\psi-\epsilon}{\Theta}} dp_1 \cdots dq_3 = \int_{-\infty}^{+\infty} \cdots \int \frac{\psi-\epsilon}{\Theta} e^{\frac{\psi-\epsilon}{\Theta}} dp_1 \cdots dq_3$$

or

$$\frac{\partial \psi}{\partial \Theta} \cdot 1 = \int_{-\infty}^{+\infty} \cdots \int \eta e^{\eta} dp_1 \cdots dq_3 = \bar{\eta}.$$

That the factor $\partial\psi/\partial\,\Theta$ may be transferred to outside the integral sign depends on the fact that ψ in (3) does not contain the variables with regard to which the integration is performed.

By differentiating ψ with respect to Θ the value of $\bar{\eta}$ is

$$\bar{\eta} = -\,2\log v - 3\log\Theta - 3. \tag{4}$$

And now the average value $\bar{\epsilon}$ of the energy ϵ may be found very simply. For as ψ and Θ are constants, it follows from

$$\eta = \frac{\psi - \epsilon}{\Theta} \quad \text{that} \quad \bar{\eta} = \frac{\psi - \bar{\epsilon}}{\Theta}.$$

Hence
$$\bar{\epsilon} = \psi - \Theta\,\bar{\eta} = 3\Theta. \tag{5}$$

Moreover if there were n gongs, the results would be

$$\bar{\eta} = -\,2\log v - n\log\Theta - n, \tag{4'}$$

$$\epsilon = \psi - \Theta\,\bar{\eta} = n\,\Theta. \tag{5'}$$

The parameter v which enters into the expression for the potential energy and which determines the length of the uppermost string may be regarded as an external coordinate of the system. The difference between an external and an internal coordinate is simply that the internal coordinates are those which take an active part in the motion, and in the solution are to be expressed in terms of the time; whereas the external coordinates fix the system as a whole and take no active part in the motion, although different motions will arise from varying such coordinates. Now the forces are derivable from the potential energy and they will be of two sorts, internal and external. If $Q_1, Q_2, Q_3, \cdots$ are the internal forces and A is the external force,* then

$$Q_1 = -\frac{\partial\epsilon_\eta}{\partial q_1}, \quad Q_2 = -\frac{\partial\epsilon_\eta}{\partial q_2}, \cdots, \quad A = -\frac{\partial\epsilon_\eta}{\partial v}.$$

* The external force A should not be confused with the force which is required to support the system of gongs from falling under the influence of gravity. The force of gravity does not enter essentially into our problem. In fact it might be convenient to imagine that the force of gravity were continuously diminished toward zero as a limit as if the system were removed to a point indefinitely near to center of the earth. It may be noted, however, that the force of gravity does no work on the system even when the length of the uppermost string is changed: for the assumption has been made that the wheels which pinch the string are moved up and down to regulate the length rather than that the string should be paid out between them in such a way as to raise or lower the system.

It is easy to see without recourse to formulas that if the uppermost string is lengthened, the system loses energy: for a given angular twist upon a short string represents a greater amount of energy stored in the string than does the same twist in a longer string.

The average value of the external force A over the ensemble may readily be computed. It is merely

$$\bar{A} = \int_{-\infty}^{+\infty} \cdots \int -\frac{\partial \epsilon_q}{\partial v} e^{\frac{\psi - \epsilon}{\Theta}} dp_1 \cdots dq_3.$$

To evaluate this the device of differentiating (2) will again be used, but in this case the differentiation will be with respect to v. Note that ψ contains v but Θ does not, and that as ϵ_p does not contain v, the derivatives of ϵ and ϵ_q with respect to v are equal. Then

$$\int_{-\infty}^{+\infty} \cdots \int \frac{1}{\Theta}\left(\frac{\partial \psi}{\partial v} - \frac{\partial \epsilon_q}{\partial v}\right) e^{\frac{\psi - \epsilon}{\Theta}} dp_1 \cdots dq_3 = 0,$$

$$\bar{A} = \int_{-\infty}^{+\infty} \cdots \int -\frac{\partial \epsilon_q}{\partial v} e^{\eta} dp_1 \cdots dq_3 = -\frac{\partial \psi}{dv} \int_{-\infty}^{+\infty} \cdots \int e^{\eta} dp_1 \cdots dq_3.$$

Hence
$$\bar{A} = -\frac{\partial \psi}{\partial v} \cdot 1 = 2\frac{\Theta}{v}. \tag{6}$$

The result is independent of the number of gongs.

2. A thermodynamic analogy. Let p, V, T, U, S be respectively the pressure, volume, temperature, energy, and entropy of a gas. Let the gas be assumed to be in so far perfect or ideal as to satisfy Boyle's characteristic equation $pV = RT$ without sensible error. Further let the units of measurement be those ordinarily adopted in theoretical and practical physical chemistry.* Then $pV = 2T$ is the form of the characteristic equation. Moreover the energy of a perfect gas varies directly with the absolute temperature and is $U = c_v T$ where c_v denotes the specific heat at constant volume. In the

* At the absolute temperature $T = 273°$ centigrade of melting ice and under the normal atmospheric pressure of 76 cm. of mercury or 1033 gr. per sq. cm., the gram-molecular volume is 22,380 cu. cm. Hence

$$R = \frac{1033 \times 22380}{273 \times 42350} = 2 \quad \text{(or just a trifle less).}$$

See Walker's *Introduction to Physical Chemistry*, 2nd edition, p. 30, or any similar work.

physico-chemical system of units $c_v = 3$ for a monatomic gas; for the ordinary diatomic gas $c_v = 5$; and for more complicated gases the values are 6, 7, 8 or numbers in the immediate vicinity of these integers.* Hence there is the equation

$$U = c_v T = nT, \qquad n = 3, 5, 6, 7, 8,$$

for the energy of any ordinary simple perfect gas. The entropy and free energy of the gas have the familiar forms

$$S = c_v \log T + 2 \log V + \text{const.} \quad \text{and} \quad \psi = -2T \log V - c_v T \log T;$$

and finally, the thermodynamic surface for the gas, which is the equation between the energy, entropy and volume, is

$$U^{c_v} V^2 = Ke^S, \qquad K \text{ an arbitrary const.}$$

With these data for gases in mind it is easy to set up an analogy between the thermodynamics of a perfect gas and the averages obtained over a canonically distributed ensemble of vibrating dinner-gongs. The results may be exhibited most graphically in double columns.

Thermodynamics of a Perfect Gas	*Statistical Mechanics of the Dinner-gong*
temperature T	constant (modulus) Θ
energy U	average energy $\bar{\epsilon}$
entropy S	negative average index of probability $-\bar{\eta}$
free energy ψ	constant ψ
volume V	external coordinate $v \propto l^3$
pressure $p = -\left(\frac{\partial \psi}{\partial V}\right)_T$	average external generalized force $\bar{A} = -\left(\frac{\partial \psi}{\partial v}\right)_\Theta$
specific heat c_v	degrees of freedom n
equation $pV = 2T$ †	equation $\bar{A}v = 2\Theta$

* These numbers are of course the "degrees of freedom" of the molecule. Chlorine is a somewhat anomolous diatomic gas with $c_v = 6$. Ethylene give $c_v = 8$.

† This is the so-called characteristic equation of Boyle for perfect gases.

equation $U = c_v T$	equation $\bar{\epsilon} = n\Theta$
equation $\psi = -2T\log V - c_v T\log T$	equation $\psi = -2\Theta\log v - n\Theta\log\Theta$
equation $S = c_v \log T + 2\log V$ + const.*	equation $-\bar{\eta} = n\log\Theta + 2\log v + n$
equation $U^{c_v} V^2 = Ke^S$†	equation $\bar{\epsilon}^n v^2 = k\, e^{-\bar{\eta}}$

It therefore appears that the statistical mechanics of the system containing n gongs has all the formal properties of the thermodynamics of a perfect gas except that the arbitrary constant which enters into the entropy of a gas (and consequently into certain other formulas) is determined in the dynamical system. It may be well to note in particular some of the correspondences. The volume of the gas is related with the fourth root of the length of the uppermost string. If the volume is increased, the gas does work and loses energy; if the string is lengthened energy is lost to the system. To the specific heat corresponds the number of gongs, that is, the number of degrees of freedom in the system—and this is precisely the number of degrees of freedom assigned to the molecules of the gas. As the above equations contain the basis of all the thermodynamics of the gas and have identical analogs in the averages taken over the canonically distributed ensemble of dynamical systems, it appears that the development of the two could never diverge. This conclusion, however, is a bit hasty for the reason that nothing has been developed which is analogous to the mixing of two gases. As a matter of fact the mixing of gases depends on a principle in addition to those here given for a simple gas and contains a somewhat troublesome paradox known as Gibbs's paradox. The matter of mixing gases and the analogous combinations of dynamical systems will therefore be omitted from the present discussion.

There is one point of general import which should be mentioned. The reason for replacing the study of one particular vibrating system of gongs by the study of average values over an ensemble was that nothing was supposed to be known concerning the initial conditions of any one system (page

* The entropy necessarily contains an additive constant as it is defined by an integral for which there is no natural lower limit — the absolute zero of temperature being excluded because the integral diverges for $T = 0$. Similarly $\psi = U - TS$ contains an arbitrary constant and may be written $\psi = -2T\log V - c_v T\log T + (c_v - \text{const.})\, T$; but the origin of entropy may be so chosen as to make this final term vanish. The constant n which occurs in $-\bar{\eta}$ is not arbitrary.

† The equation of the thermodynamic surface. The constant K is arbitrary owing to the additive constant in S; but $k = e^{-n}$.

137–8); the only known fact was the total energy of the system and the differential equations of motion immediately derivable therefrom. The question naturally arises: How nearly do the average values represent the conditions likely to obtain in a system selected at random? This is quite like the analogous question in the theory of errors of observation: How reliable is the result as determined by the theory? A measure of the reliability of the result is assumed to be found in the value of the mean square error; and for the dynamical systems the average value of the square of the departure $\epsilon - \bar{\epsilon}$ of the energy from its mean value may naturally be taken as a measure of how far the average state differs from the state of a system selected at random. It remains, therefore to evaluate the average of $(\bar{\epsilon} - \epsilon)^2$.

Now it is clear that the relations

$$\overline{(\epsilon - \bar{\epsilon})^2} = \overline{\epsilon^2 - 2\epsilon\bar{\epsilon} + \bar{\epsilon}^2} = \bar{\epsilon}^2 - 2\bar{\epsilon}\,\bar{\epsilon} + \bar{\epsilon}^2 = \bar{\epsilon}^2 - \bar{\epsilon}^2$$

hold for the averages. It is therefore necessary to evaluate $\bar{\epsilon}^2$; and the device of differentiating under the sign of integration may again be advantageously employed. By definition and by (5) we have

$$\bar{\epsilon} = \int_{-\infty}^{+\infty} \cdots \int \epsilon e^{\frac{\psi - \epsilon}{\Theta}}\, dp_1 \cdots dq_3 = 3\Theta.$$

Now differentiate with respect to Θ on both sides. Then

$$\int_{-\infty}^{+\infty} \cdots \int -\epsilon \left[\frac{\psi - \epsilon}{\Theta^2} - \frac{1}{\Theta}\frac{\partial \psi}{\partial \Theta}\right] e^{\frac{\psi - \epsilon}{\Theta}}\, dp_1 \cdots dq_3 = 3.$$

or
$$\int_{-\infty}^{+\infty} \cdots \int \epsilon^2 e^{\frac{\psi - \epsilon}{\Theta}}\, dp_1 \cdots dq_3 - \psi \int_{-\infty}^{+\infty} \cdots \int \epsilon e^{\frac{\psi - \epsilon}{\Theta}}\, dp_1 \cdots dq_3 + \Theta \frac{\partial \psi}{\partial \Theta} \int_{-\infty}^{+\infty} \cdots \int \epsilon e^{\frac{\psi - \epsilon}{\Theta}}\, dp_1 \cdots dq_3 = 3\Theta^2.$$

or
$$\bar{\epsilon}^2 - \psi\,\bar{\epsilon} + \Theta \frac{\partial \psi}{\partial \Theta}\,\bar{\epsilon} = 3\Theta^2.$$

The substitution from (5) and (3) or from (5′) or (3′) gives

$$\bar{\epsilon}^2 = 12\Theta^2 \quad \text{or} \quad n(n+1)\Theta^2; \quad \bar{\epsilon}^2 - \bar{\epsilon}^2 = 3\Theta^2 \quad \text{or} \quad n\Theta^2,$$

according as the number of gongs is 3 or n. Hence the ratio

$$\overline{(\epsilon - \bar{\epsilon})^2} : \bar{\epsilon}^2 = 1 : 3 \quad \text{or} \quad 1 : n.$$

It therefore appears that as far as the energy is concerned, the mean square error in assuming that the average represents a system selected at random is considerable when n is small but not very large when n is great and becomes quite negligible when n is very great. A similar calculation could be carried out for the quantities

$$\overline{(\eta - \bar{\eta})^2} \quad \text{and} \quad \overline{(A - \bar{A})^2};$$

but this will be left to the reader.

A simple device will give a system in which the average condition better represents the condition probably found in a system selected at random. For suppose that instead of considering single sets of n gongs, the attention is turned toward pairs of exactly similar sets or in general to a system consisting of N of the single systems of gongs. This compound system will have Nn degrees of freedom if each individual set has n gongs. The energy of the compound system will be

$$\epsilon = \sum_1^N \epsilon_i = \pi \sum_1^N$$

$$\left[a_1 p_{1i}^2 + p_{2i}^2 + p_{3i}^2 + \cdots + v^{-4} q_{1i}^2 + (q_{2i} - q_{1i})^2 + (q_{3i} - q_{2i})^2 + \cdots \right].$$

If the compound systems are to be canonically distributed over their representative space of $2Nn$ dimensions, the probability will be $e^{\frac{\psi - \epsilon}{\Theta}}$ where ϵ has this new value. As there are no relations between the different sets of coordinates and momenta $p_{1i}, p_{2i}, p_{3i}, \cdots, q_{1i}, q_{2i}, q_{3i}, \cdots$, it is readily seen that the quantities connected with the compound systems are

$$\psi = -2N\Theta \log v - Nn\,\Theta \log \Theta$$

$$\bar{\eta} = \frac{\partial \psi}{\partial \Theta} = -2N \log v - Nn \log \Theta - Nn$$

$$\bar{\epsilon} = \psi - \Theta\bar{\eta} = Nn\Theta$$

$$\bar{A} = -\frac{\partial \psi}{\partial v} = 2N\frac{\Theta}{v}, \qquad \bar{A}v = 2N\Theta$$

$$\overline{(\epsilon - \bar{\epsilon})^2} : \bar{\epsilon}^2 = 1 : Nn$$

By taking N sufficiently large, the probability of the energy of an arbitrarily selected system being noticeably at variance with the average energy is very small. Moreover the only difference between the new and the old equations is that ψ, $\bar{\eta}$, $\bar{\epsilon}$, $\bar{A}$, are multiplied by the constant N — a difference which is of the same sort as if these quantities were measured in different units. The thermodynamic analogy therefore persists in all its essentials. It even has an additional advantage in that now any one system represents the average or vice versa with great probable precision, which is surely true of any particular sample of gas and the average of all similar samples.

3. Microcanonical ensembles. A microcanonical distribution is derived from the canonical by confining the attention to those systems of the ensemble which are characterized by representative points between the surfaces $\epsilon = C$ and $\epsilon + d\epsilon = C + dC$ in the representative space. Now as the probability depends only on the energy and the energy is sensibly constant between these two surfaces, it follows that the actual infinitesimal probability (see page 134) that the representative point shall lie between the two successive surfaces is

$$P dV = e^{\frac{\psi - \epsilon}{\Theta}} \, dV = e^{\frac{\psi - \epsilon}{\Theta}} \left(\frac{dv}{d\epsilon}\right)^*_v d\epsilon,$$

where V represents the volume of the six-dimensional space included by the surface $\epsilon = C$ —a quadric surface in $p_1, p_2, p_3, q_1, q_2, q_3$, —and dV represents the increment of volume between this surface and the succeeding one.

It is necessary to calculate the volume

$$V = \int_{\epsilon=0}^{\epsilon=C} \cdots \int dp_1 dp_2 dp_3 dq_1 dq_2 dq_3$$

within the ellipsoid $\epsilon = C$, where for definiteness and simplicity the problem has been restricted to the case of the single system consisting of only three gongs which is the case first handled in the previous discussion. The easiest way of evaluating V is to make the change of variable

$$x_1 = \sqrt{\pi a_1} p_1, \quad x_2 = \sqrt{\pi a_2} p_2, \quad x_3 = \sqrt{\pi a_3} p_3, \quad x_4 = \sqrt{\pi} v^{-2} q_1, \quad x_5 = \sqrt{\pi}(q_2 - q_1)$$

$$x_6 = \sqrt{\pi}(q_3 - q_2), \quad dp_1 \cdots dq_3 = \frac{v^2}{\pi^3} dx_1 \cdots dx_6.\dagger$$

* As a system with a definite length for the uppermost string is under discussion, the quantity v is supposed to be constant at this point; later it will be allowed to vary.

† It may be recalled that to find the new element of volume when changing the variable in a multiple integral the Jacobian is introduced.

The integral to be evaluated then takes the form

$$V = \frac{1}{\pi^3 v^2} \int_{\epsilon=0}^{\epsilon=C} \cdots \int dx_1 dx_2 dx_3 dx_4 dx_5 dx_6,$$

where $$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 \leqq C.$$

This is a classic integral evaluable by means of the Γ– functions. The result is *

$$V = \frac{v^2}{\pi^3} \frac{[\Gamma(\tfrac{1}{2})]^6}{\Gamma(3)} \int_0^C h^2 dh = \tfrac{1}{6}\ C^3 v^3 = \tfrac{1}{6}\ \epsilon^3 v^2. \tag{7}$$

Now the expression for the probability may be written

$$P dV = e^{\frac{\psi - \epsilon}{\Theta}} \cdot \frac{1}{2} \frac{\epsilon^2}{v^2}\, d\epsilon\,;$$

and may be given a more convenient form by letting

$$\phi = \log\left(\frac{dV}{d\epsilon}\right)_v = 2\log\epsilon - 2\log v - \log 2,$$

so that $$P dV = e^{\frac{\psi - \epsilon}{\Theta} + \phi}\, d\epsilon \quad \text{and} \quad \int_0^\infty e^{\frac{\psi - \epsilon}{\Theta} + \phi}\, d\epsilon = 1,$$

the last condition being the new form of (2).

Now a second thermodynamic analogy may be set up. This time the analogy is between the quantities arising in the theory of a perfect gas and the average taken over a microcanonical ensemble. Equation (7) may be written as

$$\epsilon^3 v^2 = k e^{\log V} \qquad \text{and compared with} \qquad U^3 V^2 = K e^S.$$

It is then seen that the following are analogical:

In the Ensemble	*In the Gas*
the energy ϵ	the energy U
the quantity v	the volume V
the quantity $\log V$	the entropy S

* See Todhunter's *Integral Calculus*, p. 263, article 278.

The list may be further extended by the use of the fundamental thermodynamic formula

$$d\epsilon = dH - pdV = TdS - pdV$$

compared with the result of differentiating ϵ, namely,

$$d\epsilon = \left(\frac{\partial \epsilon}{\partial \log V}\right)_v d\log V + \left(\frac{\partial \epsilon}{\partial v}\right)_{\log V} dv = e^{-\phi} V d\log V + \left(\frac{\partial \epsilon}{\partial v}\right)_{\log V} dv.$$

Then the following correspondences may be set up:

In the Ensemble	*In the Gas*
$\left(\frac{\partial \epsilon}{\partial \log V}\right)_v = e^{-\phi} V = \frac{1}{3}\epsilon$	the temperature T
$-\left(\frac{\partial \epsilon}{\partial v}\right)_{\log V} = \frac{2\epsilon}{3v} = A$	the pressure p
Equation $Av = 2e^{-\phi} V$	Equation $pV = 2T$

There seems to be no need to carry the table further. Similar results could have been found for the system with n gongs.

This second analogy has the distinct advantage that the energy of the dynamical system corresponds to the energy of the gas, whereas in the earlier analogy it was the average energy over the whole ensemble which corresponded to the energy of the gas. This is not the place and the necessary fine distinctions between closely related ideas in the theory of probability have not been sufficiently set forth to enter profitably into the discussion of the relative merits of the two analogies. On the whole, physicists seem to prefer the second, apparently mainly for the reason that the correspondence of the energies seems to them more natural; but Gibbs evidently inclined rather to prefer the first and to feel that (in view perhaps of the facts that for systems of so many degrees of freedom as a gas the energy of any one system could scarcely be expected to differ appreciably from the average energy and that we really do not know what the energy of the gas as a dynamical system is except in so far as it makes itself felt through the sort of average phenomena of pressure and temperature) the first correspondence was quite satisfactory; there are, moreover, some other reasons why it is more satisfactory.

Conclusion. The foregoing simple dynamical system in which the computations of finding averages have been so easy as to be carried out without difficulty and for which the thermodynamic analogies are immediately comparable with the ordinary theory of gases has been treated as far as possible in parallel with the general discussion of statistical mechanics given by J. Willard Gibbs in his *Elementary Principles in Statistical Mechanics*. That the simple and concrete case may perhaps be useful as an introduction to the general and abstract presentation of the statistical mechanics of any system which satisfies the Hamiltonian equations of motion is its only excuse for existence. There are, however, a number of matters which suggest themselves in connection with the foregoing work and with statistical mechanics in general which may deserve a moment's notice.

In the first place it might be considered remarkable that a system so simple and so apparently unlike a gas should give the thermodynamical equations of a gas. A reference to Gibbs's work will show, however, that the fundamental differential equation of thermodynamics and the fundamental equation of statistical mechanics

$$dU = TdS - pdV \quad \text{and} \quad d\bar{\epsilon} = \Theta d(-\bar{\eta}) - \bar{A}dV \quad \text{or} \quad d\epsilon = \epsilon^{-\phi} Vd\log V - Adv,$$

whether in the canonical or microcanonical ensemble, will always correspond when there is only one variable external coordinate in the system. It therefore only remains to select the dynamical system in such a way as to give the proper special results for a gas, namely, the characteristic equation, the free energy, and so on. The selection was easy, especially in view of the general law of equipartition of kinetic energy.

A few words may be said in explanation of this famous law propounded by Maxwell and discussed by practically every student of the theory of gases since that time. In the case in hand, the kinetic energy is a sum $\pi(a_1 p_1^2 + a_2 p_2^2 + a_3 p_3^2)$ of the squares of the momenta and it is clear that the individual terms $\pi a_1 p_1^2$, $\pi a_2 p_2^2$, $\pi a_3 p_3^2$ represent the kinetic energies of the individual gongs. The average value of these partial kinetic energies is easily found. For instance

$$\overline{\pi a_1 p_1^2} = \frac{\int_{-\infty}^{+\infty} \cdots \int \pi a_1 p_1^2 e^{\frac{\psi - \epsilon_p - \epsilon_q}{\Theta}} dp_1 \cdots dq_3}{\int \cdots \int e^{\frac{\psi - \epsilon_p - \epsilon_q}{\Theta}} dp_1 \cdots dq_3}$$

$$= \frac{\int_{-\infty}^{+\infty} \pi a_1 p_1^2 e^{\pi a_1 p_1^2} dp_1 \int \cdots \int e^{\frac{\psi - \pi a_2 p_2^2 - \pi a_3 p_3^2 - \epsilon_q}{\Theta}} dp_2 \cdots dq_3}{\int_{-\infty}^{+\infty} e^{\pi a_1 p_1^2} dp_1 \int \cdots \int e^{\frac{\psi - \pi a_2 p_2^2 - \pi a_3 p_3^2 - \epsilon_q}{\Theta}} dp_2 \cdots dq_3}.$$

Now the integrals though properly iterated, may be considered as merely multiplied: for p_1 does not occur in the last five provided the limits are not taken as functions of p_1. Hence the last five integrals may be canceled from numerator and denominator and the calculation of the simple integrals remaining gives

$$\overline{\pi a_1 p_1^2} = \tfrac{1}{2}\,\Theta.$$

It therefore is seen that the average of each partial kinetic energy corresponding to each individual gong is $\frac{1}{2}\,\Theta$ and the average value of the total kinetic energy is $\frac{1}{2}\,3\Theta$ or $\frac{1}{2}n\Theta$. The fact that each gong has the same average kinetic energy as every other constitutes the law of equipartition of kinetic energy for this particular case. With the proper conventions the same result is obtained for any dynamical system.

The proof that the average kinetic energy in a system of n degrees of freedom is $\frac{1}{2}\,n\,\Theta$ and that each degree of freedom has its aliquot share of the whole depends, as Gibbs gives it, on the assumption of the canonical distribution including the hypothesis that the momenta may have any values between $-\infty$ and $+\infty$ and that the limits of the integrations with respect to the coordinates $q_1, q_2, \cdots, q_n$ do not depend on the momenta. If any of these assumptions were untrue, the proof would not hold. The assumptions seem ideally simple with the exception of the initial assumption of canonical distribution; and at any rate the fact of the at least approximate truth of the law of equipartition of energy is so strikingly confirmed in the theory of gases as to justify the detailed discussion of any reasonable theory which has it as a consequence. This will perhaps serve as an answer to those who might inquire why the particular law of density or probability for the ensemble known as the canonical law was chosen for the work of this article; there are moreover numerous more or less accurate and plausible "proofs" of the law or reasons

for it which may be found in various treatises on the theory of gases and in Gibbs's book; but they cannot be taken up here. Other assumptions for the density or probability would have enabled us to compute averages but would in all probability have been destitute of thermodynamic analogy.

Although the general theory of statistical mechanics reached a high development at the hands of Gibbs, the detailed application of the method of investigation to systems that are of thermodynamic importance in physics has scarcely been touched; special methods and approximations for obtaining averages, under the initial assumption not of the canonical distribution but of the law of equipartition of energy, have generally been employed. Indeed it is readily seen that to evaluate an iterated integral like

$$\int_{-\infty}^{+\infty} \cdots \int dp_1 \cdots dp_n \int \cdots \int e^{\frac{\psi - \epsilon_p - \epsilon_q}{\Theta}} dq_1 \cdots dq_n$$

(even where the limits for the integrations with respect to the q's are taken as independent of the p's) is out of the question for most sets of limits and for most assumptions for the potential energy ϵ_q even if the simple assumption be made that the kinetic energy consists of a sum $\Sigma a_i p_i^2$ with constant coefficients. For instance if x_i, y_i, z_i and x_j, y_j, z_j are the coordinates of the centers of two monatomic molecules of a gas, and if the usual law of attraction inversely as the square of the distance and repulsion as the inverse fifth power of the distance be assumed, the potential energy would take the form

$$\epsilon_q = \sum_{i,j} \left[\frac{-a}{\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}} + \frac{b}{\left[(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2\right]^2} \right],$$

to which other terms must be added to keep the molecules within the vessel containing the gas.

Despite the difficulties which seem to lie in the way of the systematic practical application of the statistical mechanics of a canonical distribution of dynamical systems, the fact that the theory leads to the fundamental differential relation of thermodynamics will probably lead to its detailed investigation. It is hoped that the presentation here given may be an aid to those who wish to make a start on this field of science.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
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DISCUSSION OF A METHOD FOR FINDING NUMERICAL SQUARE ROOTS

By C. L. Bouton

In a recent article* Professor E. R. Hedrick gave a method for approximating to the positive square root of any positive number. Inasmuch as he merely states the method without giving proofs, and as the proof itself is a rather pretty exercise in the elementary theory of limits, it seems worth while to give a discussion of the method. The facts that the method is really Newton's method of approximating to the square root, and that it may easily be extended to the p^{th} root, are of interest. Professor Hedrick's plea that this method be substituted for the method of extracting square root ordinarily used seems to be scarcely justified on the ground of rapidity of convergence, although there may be pedagogical advantages in its use with young pupils. The method has the further disadvantage that when applied to a number which is a perfect square, it yields the square root only as the result of an infinite process, unless indeed the *first* trial root should turn out to be the exact root; whereas, the ordinary process applied to this case always terminates in a finite number of steps, provided that the given number is an integer, or a finite decimal fraction.

The process is the following: Let a be the given number, select any positive number n, and form in succession

$$(1)\qquad \begin{cases} n_1 = \frac{1}{2}\left(n + \frac{a}{n}\right), \\ n_2 = \frac{1}{2}\left(n_1 + \frac{a}{n_1}\right), \\ \quad \cdots\cdots\cdots \\ n_i = \frac{1}{2}\left(n_{i-1} + \frac{a}{n_{i-1}}\right). \end{cases}$$

Then we shall prove that the numbers of the sequence $n_1, n_2, \cdots, n_i$, form a decreasing set, and approach $\sqrt{a}$ as their limit. The process converges

* *Western Journal of Education*, vol. I, October, 1908, p. 252.

rapidly enough to be useful in computing numerical square roots. For example, let $a = 2$, and choose $n = 1$. Then we compute in succession

$$n_1 = 1.5, \quad n_2 = 1.417, \quad n_3 = 1.414216$$

Thus in three steps the error is about 2 in 10^6.

To prove that $\lim_{i=\infty} (n_i) = \sqrt{a}$, and to estimate the error at any step, let us proceed as follows: First we have the identity

$$(2) \qquad n_i^2 - a = \frac{1}{4}\left(n_{i-1} + \frac{a}{n_{i-1}}\right)^2 - a = \left(\frac{n_{i-1}^2 - a}{2n_{i-1}}\right)^2 \geqq 0,$$

whence

$$(3) \qquad n_i \geqq \sqrt{a} \qquad (i = 1, 2, \cdots).$$

That is, $n_1, n_2, \cdots$ are all greater than $\sqrt{a}$, unless n was chosen equal to $\sqrt{a}$, in which case $n_1 = n_2 = \cdots = \sqrt{a}$. From this point on we shall assume that n was not chosen equal to $\sqrt{a}$, so that in (2) and (3) the equality sign cannot hold. Moreover, from (1) and (3) we have

$$(4) \qquad n_{i-1} - n_i = \frac{n_{i-1}^2 - a}{2n_{i-1}} > 0 \qquad (i = 2, 3, \cdots),$$

so that

$$(5) \qquad n_i < n_{i-1} \qquad (i = 2, 3, \cdots).$$

Therefore the variable n_i always decreases when i increases, but n_i is always greater than $\sqrt{a}$. Hence by a fundamental principle in the theory of limits* the limit of n_i exists when $i = \infty$; let us call it N. We readily see that this limit is $\sqrt{a}$, for from (2) and (4) we have

$$(6) \quad n_i - \sqrt{a} = \frac{1}{n_i + \sqrt{a}}\left(\frac{n_{i-1}^2 - a}{2n_{i-1}}\right)^2 = \frac{(n_{i-1} - n_i)^2}{n_i + \sqrt{a}} \qquad (i = 1, 2, \cdots).$$

Passing to the limit for $i = \infty$, we get

$$N - \sqrt{a} = \frac{(N - N)^2}{N + \sqrt{a}} = 0,$$

$$\therefore \qquad N = \sqrt{a}.$$

* See, for example, Osgood, *Introduction to Infinite Series*, p. 4.

This proves the convergence of the process. It is easy to set limits to the error at any step, and to form some idea of the rapidity of the convergence. We have from (6) and (3)

$$(7) \qquad \frac{(n_{i-1}-n_i)^2}{2n_i} < n_i - \sqrt{a} < \frac{(n_{i-1}-n_i)^2}{2\sqrt{a}} < \frac{n_i(n_{i-1}-n_i)^2}{2a}.$$

That is, if we set $n_{i-1} - n_i = \delta_i$ and $n_i - \sqrt{a} = \epsilon_i$, then

$$(8) \qquad \frac{\delta_i^2}{2n_i} < \epsilon_i < \frac{n_i\delta_i^2}{2a} \qquad (i = 1, 2, \cdots).$$

Furthermore, from (7) and (3)

$$(9) \qquad \frac{n_i - \sqrt{a}}{\sqrt{a}} < \frac{(n_{i-1}-n_i)^2}{2a} < \frac{1}{2}\left(\frac{n_{i-1}-\sqrt{a}}{\sqrt{a}}\right)^2.$$

Let $\dfrac{n_i - \sqrt{a}}{\sqrt{a}} = r_i$, so that r_i is the relative or percentage error at the i^{th} step. Then

$$(10) \qquad r_i < \tfrac{1}{2} r_{i-1}^2 \qquad (i = 2, 3, \cdots).$$

Repeated application of (10) shows that

$$(11) \qquad r_i < \frac{1}{2^{2^i-1}}\left[\frac{(n^2-a)^2}{n^2a}\right]^{2^i-1} \qquad (i = 1, 2, \cdots).$$

Formula (8) shows that the actual error of any approximation is nearly proportional to the square of the difference between that approximation and the preceding one. Formula (10) gives the relation between the *relative* error at any step, and that of the preceding step. It shows that when working decimally the number of significant figures retained at any step should be double the number which are correct at the preceding step. These formulae both show that while the process may not converge very rapidly at first, it later converges with extreme rapidity. Thus in the computation for $\sqrt{2}$ we found $r_3 < 2 \times 10^{-6}$, hence $r_4 < 2 \times 10^{-12}$ and $r_5 < 2 \times 10^{-24}$.

Formula (11) enables us to estimate the number of steps necessary to obtain a given degree of approximation, and furthermore shows the effect of the choice of the initial n. Of course the nearer n is chosen to $\sqrt{a}$ the more

rapid is the convergence. The method is therefore particularly useful for computing square roots when an approximate value is known, as for example, in extending a given table of square roots to a larger number of places. Thus, if we take $a = 3$, and choose $n = 1.732$, a single step gives $n_1 = 1.732050808314$. If to this we apply the correction computed from (8) we find 1.732050807569, while the value of $\sqrt{3}$ correct to thirteen decimal places is 1.7320508075689.

Furthermore, it should be noticed that the commission of an arithmetical error at *any* stage of the work does not vitiate the ultimate result, for it simply amounts to starting over with a new value of n. In fact, such a mistake may even *increase* the rapidity of the convergence.

A geometrical proof of the convergency of the process is easily given. Draw the rectangular hyperbola whose equation is

$$xy = a.$$

On this hyperbola connect the points $\left(n, \frac{a}{n}\right)$ and $\left(\frac{a}{n}, n\right)$. Let C_1 be the

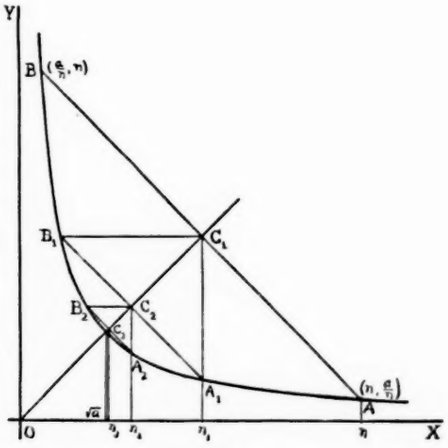

FIG. 1.

middle point of this chord. It is easy to see that the coordinates of C_1 are (n_1, n_1). Then repeat the process with n_1 in place of n, and so on indefinitely. An examination of the figure shows at once that the points C_1, C_2, $C_3 \cdots$ are moving down the diagonal towards the vertex, but that they never go beyond the vertex. Hence the limit of n_i is $\sqrt{a}$. This figure further shows that

the value of n assumed at the start has no influence on the final result, but that if a value of n near $\sqrt{a}$ be chosen the process will converge much more rapidly than if we begin with a value remote from $\sqrt{a}$.

It is interesting to apply this geometrical method to trying to find the square root of a negative number. The process does not converge.

Finally, it should be noticed that this process is exactly that given by Newton's method for approximating to the positive root of the equation

$$x^2 - a = 0.$$

In that method, the curve

$$y = x^2 - a$$

is drawn, any value of x is selected, the tangent is drawn at the corresponding

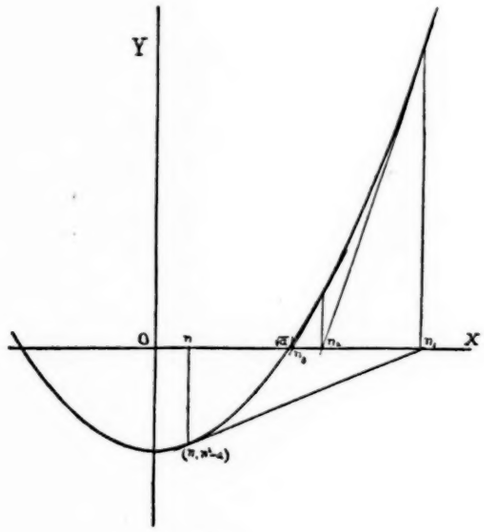

FIG. 2.

point of the curve, and its intersection with the X-axis gives a new value of x, with which the process is repeated. In this particular case the equation of the tangent at $(n, n^2 - a)$ is

$$y - (n^2 - a) = 2n(x - n)$$

or
$$2nx = y + n^2 + a$$

whence
$$n_1 = \frac{1}{2}\left(n + \frac{a}{n}\right), \text{ if } n_1 \text{ be the } X\text{-intercept};$$

and the process is repeated. But this is exactly the method described above.

Newton's method applied to the p^{th} root gives at once

$$(12) \qquad \begin{cases} n_1 = \dfrac{1}{p}\left[(p-1)n + \dfrac{a}{n^{p-1}}\right], \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ n_i = \dfrac{1}{p}\left[(p-1)n_{i-1} + \dfrac{a}{n_{i-1}^{p-1}}\right]. \end{cases}$$

If we write

$$n_{i-1} - n_i = \delta_i, \quad n_i - \sqrt[p]{a} = \epsilon_i, \quad \frac{\epsilon_i}{\sqrt[p]{a}} = r_i$$

it is easy to prove from (12) that

$$\frac{(p-1)\delta_i^2}{2n_i} < \epsilon_i < \frac{(p-1)\delta_i^2}{2[n_i - (p-1)\delta_i]},$$

and
$$r_i < \frac{p-1}{2} r_{i-1}^2 (1 + r_{i-1})^{p-2}.$$

These formulae are direct generalizations of (8) and (10), for the actual and relative error of any approximation.

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ON THE DIRECT PRODUCT IN THE THEORY OF FINITE GROUPS*

By J. H. Maclagan-Wedderburn

1. This paper is an investigation into the formation of groups by the process known as the "direct product." The calculus introduced by Frobenius is used throughout with the addition of the sign $\mathbf{A} \frown \mathbf{B}$ to denote the largest sub-complex which is common to the complexes $\mathbf{A}$ and $\mathbf{B}$; simple isomorphism between $\mathbf{A}$ and $\mathbf{B}$ is denoted by $\mathbf{A} \sim \mathbf{B}$.

Groups are represented by Gothic letters and their elements by the corresponding Roman capitals; ordinary numbers are represented by small Roman letters, and in particular the order of a group is denoted by the same letter of the alphabet as the group itself. Thus the elements of a group $\mathbf{A}$ are denoted by $A_1, A_2, \cdots A_a$, a being its order. The identity is denoted as usual by I.

2. If a group $\mathbf{H}$ has two sub-groups $\mathbf{A}$ and $\mathbf{B}$ such that (1) $\mathbf{H} = \mathbf{AB}$, (2) $\mathbf{A} \frown \mathbf{B} = I$, (3) every element of $\mathbf{A}$ is commutative with every element of $\mathbf{B}$, then $\mathbf{H}$ is said to be the *direct product* of $\mathbf{A}$ and $\mathbf{B}$, and $\mathbf{A}$ and $\mathbf{B}$ are called *direct factors* of $\mathbf{H}$.

The relation between $\mathbf{H}$, $\mathbf{A}$ and $\mathbf{B}$ is conveniently† denoted by $\mathbf{H} = \mathbf{A} \times \mathbf{B}$. Where the meaning is otherwise clear, the adjective "direct" may be omitted and $\mathbf{A}$ and $\mathbf{B}$ simply called factors of $\mathbf{H}$. In this paper the term "factor" will be used solely in this sense. A group whose only factors are itself and the identity is said to be *prime*.

It may be noticed here that, instead of saying that every element of $\mathbf{A}$ is commutative with every element of $\mathbf{B}$, it is sufficient to state that $\mathbf{A}$ and $\mathbf{B}$ are invariant.‡

3. If $\mathbf{G}$ is any subgroup of $\mathbf{H} = \mathbf{A} \times \mathbf{B}$, every element of $\mathbf{G}$ can be expressed in the form $G_r = A_{a_r} B_{b_r}$. The two sets A_{a_r} and B_{b_r} $(r = 1, 2, \cdots g)$ form subgroups of $\mathbf{A}$ and $\mathbf{B}$ respectively which are isomorphic with $\mathbf{G}$. These

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* Read before the American Mathematical Society, September 10, 1908.

† Cf. Hurwitz, *Mathematische Annalen*, vol. 45 (1894), p. 389.

‡ Cf. Dyck, *Mathematische Annalen*, vol. 22 (1883), p. 97, and Burnside, *Theory of Groups of Finite Order* (1897), p. 44.

subgroups, which are of constant occurrence, are conveniently denoted by $\mathbf{A}_g$ and $\mathbf{B}_g$. These groups are, of course, only defined with reference to some particular method of breaking up $\mathbf{H}$ into factors. Thus if $\mathbf{H} = \mathbf{A} \times \mathbf{B} = \mathbf{A} \times \mathbf{C}$, the first factorisation gives $\mathbf{A}_b = I$, while this is not the case when the second factorisation is taken, unless $\mathbf{B} = \mathbf{C}$, which is by no means necessarily the case.

THEOREM 1. *If* $\mathbf{G}$ *is a non-abelian invariant sub-group of* $\mathbf{A} \times \mathbf{B}$, *either* $\mathbf{A} \frown \mathbf{G} > I$ *or* $\mathbf{B} \frown \mathbf{G} > I$.

If possible let $\mathbf{A} \frown \mathbf{G} = I = \mathbf{B} \frown \mathbf{G}$. Then evidently $\mathbf{A}_g \sim \mathbf{G} \sim \mathbf{B}_g$. Since $\mathbf{A}$ and $\mathbf{G}$ are invariant and $\mathbf{A} \frown \mathbf{G} = I$, every element of $\mathbf{A}$ is commutative with every element of $\mathbf{G}$. Hence $\mathbf{A}_g$ is abelian: for if $G = A_1 B$ and A_3 is any element of $\mathbf{A}$, then $A_2 A_1 B = A_2 G = G A_2 = A_1 B A_2 = A_1 A_2 B$, and therefore $A_2 A_1 = A_1 A_2$. But, since $\mathbf{G} \sim \mathbf{A}_g$ and $\mathbf{G}$ is not abelian, this is impossible; hence the theorem is valid.

THEOREM 2. *If* $\mathbf{A} \times \mathbf{B} = \mathbf{A} \times \mathbf{C}$, *then* $\mathbf{B} \sim \mathbf{C}$.

For $\mathbf{B}$ and $\mathbf{C}$ are both simply isomorphic with the quotient group of $\mathbf{A} \times \mathbf{B}$ with respect to $\mathbf{A}$. We may also notice here that $\mathbf{B}_c = \mathbf{B}$, since $\mathbf{A}$ and $\mathbf{C}$ are both contained in $\mathbf{AB}_c$.

If $\mathbf{BC} = \mathbf{B} \times \mathbf{A}'$ where $\mathbf{A}' \leqq \mathbf{A}$, it is easily shown, as in the proof of the last theorem, that $\mathbf{A}'$ is a sub-group of the central of $\mathbf{A}$. If then the central of $\mathbf{A}$ reduces to the identity, we have $\mathbf{B} = \mathbf{C}$.

4. If $\mathbf{H} = \mathbf{A}_1 \times \mathbf{A}_2 \times \cdots \times \mathbf{A}_p$ is an expression of $\mathbf{H}$ as the product of prime factors, $\mathbf{A}_1, \mathbf{A}_2, \cdots \mathbf{A}_p$ is said to be a *prime composition series* of $\mathbf{H}$. Two prime composition series are *equivalent* if they have the same number of terms and if their terms can be so arranged that corresponding ones are simply isomorphic.

THEOREM 3. *Any two prime series of a group are equivalent; and the correspondence of the terms can be so arranged that* $\mathbf{A}_b = \mathbf{A}$ *and* $\mathbf{B}_a = \mathbf{B}$, *if* $\mathbf{A}$ *and* $\mathbf{B}$ *are corresponding terms.**

Assume that the theorem is true for groups whose order is less than the order of the given group $\mathbf{H}$, and let $\mathbf{A}_1, \mathbf{A}_2, \cdots \mathbf{A}_p$ and $\mathbf{B}_1, \mathbf{B}_2, \cdots \mathbf{B}_q$ be two prime series for $\mathbf{H}$.

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* This theorem is of course well known in the case of abelian groups. In case of non-abelian groups it is implied in the known theorem that a direct product may be represented as a product of substitution groups written in distinct sets of letters. Cf. Miller, *Transactions of the Amer. Math. Society*, vol. 1 (1900), p. 71.

Let the factors be so arranged that $a_1 \leqq b_1$ and consider the group

$$\begin{aligned} \mathbf{G} &= \mathbf{A}_1 (\mathbf{B}_2 \times \cdots \times \mathbf{B}_q) \\ &= \mathbf{B}_1' \times \mathbf{B}_2 \cdots \times \mathbf{B}_q, \end{aligned}$$

where $\mathbf{B}_1' \leqq \mathbf{B}_1$.

If $\mathbf{G} = \mathbf{H}$, then $\mathbf{B}_1' = \mathbf{B}_1$, and, since $a_1 \leqq b_1$, we must have

$$\mathbf{H} = \mathbf{A}_1 \times \mathbf{B}_2 \times \cdots \times \mathbf{B}_q.$$

Theorem 2 then gives $\mathbf{A}_1 \sim \mathbf{B}_1$ and $\mathbf{A}_2 \times \cdots \times \mathbf{A}_p \sim \mathbf{B}_2 \times \cdots \times \mathbf{B}_q$, whence the first part of the theorem holds for $\mathbf{H}$ by hypothesis. The second part of the theorem obviously holds as regards $\mathbf{A}_1$ and $\mathbf{B}_1$; it can be shown to hold for the other factors as follows.

Let $\mathbf{C} = \mathbf{A}_2 \times \cdots \times \mathbf{A}_p$ and $\mathbf{D} = \mathbf{B}_2 \times \cdots \times \mathbf{B}_q$. The relations given above then become

$$\mathbf{H} = \mathbf{A}_1 \times \mathbf{C} = \mathbf{B}_1 \times \mathbf{D} = \mathbf{A}_1 \times \mathbf{D}$$

and $p = q$, $c = d$.

Now $\mathbf{A}\mathbf{C}_{b_2}\mathbf{C}_{b_3} \cdots \mathbf{C}_{b_q}$ contains $\mathbf{A}_1, \mathbf{B}_2, \cdots, \mathbf{B}_q$ and is therefore equal to $\mathbf{H}$, hence $\mathbf{C}_{b_2}^{-}\mathbf{C}_{b_3} \cdots \mathbf{C}_{b_q} = \mathbf{C}$. But $c = d = b_2 b_3 \cdots b_q$ and hence $\mathbf{C} = \mathbf{C}_{b_2} \times \mathbf{C}_{b_3} \times \cdots \times \mathbf{C}_{b_q}$, thus giving a prime series for $\mathbf{C}$. $\mathbf{C}_{b_r}$ is therefore by hypothesis simply isomorphic with say $\mathbf{A}_r$ and $(\mathbf{A}_r)_{c_{b_r}} = \mathbf{A}_r$. But $\mathbf{B}_r < \mathbf{A}_1\mathbf{C}_{b_r}$ and thence $(\mathbf{A}_r)_{c_{b_r}} = (\mathbf{A}_r)_{b_r}$. The second part of the theorem is therefore also true for $\mathbf{H}$.

Suppose now that $\mathbf{G} < \mathbf{H}$. It follows by hypothesis that $\mathbf{A}_1$ being a prime factor of $\mathbf{G}$ is simply isomorphic with $\mathbf{B}_s$ for some value of $s > 1$ and that $(\mathbf{A}_1)_{b_s} = \mathbf{A}_1$, $(\mathbf{B}_s)_{a_1} = \mathbf{B}_s$, or else that $\mathbf{A}_1 \sim \mathbf{B}_1'$.

If $\mathbf{A}_1 \sim \mathbf{B}_1'$, we have $\mathbf{A}_1 \frown \mathbf{B}_2 \times \cdots \times \mathbf{B}_q = I$, since $(\mathbf{B}_1')_{a_1} = \mathbf{B}_1'$, and therefore

$$\frac{\mathbf{H}}{\mathbf{A}_1} = \mathbf{A}_2 \times \cdots \times \mathbf{A}_p = \frac{\mathbf{B}_1}{\mathbf{B}_1'} \times \mathbf{B}_2 \times \cdots \times \mathbf{B}_q.$$

Hence, by hypothesis, to any factor $\mathbf{B}_s$ $(s > 1)$ there corresponds a factor $\mathbf{A}_s$ satisfying the conditions laid down in the theorem, viz.:

$$\mathbf{A}_s \sim \mathbf{B}_s, \quad (\mathbf{A}_s)_{b_s} = \mathbf{A}_s, \quad \text{and} \quad (\mathbf{B}_s)_{a_s} = \mathbf{B}_s.$$

Putting $s = 1$, for convenience, this gives immediately

$$\mathbf{A}_1 \times \mathbf{A}_2 \times \cdots \times \mathbf{A}_p = \mathbf{B}_1 \times \mathbf{A}_2 \times \cdots \times \mathbf{A}_p = \mathbf{B}_1 \times \mathbf{B}_2 \times \cdots \times \mathbf{B}_q,$$

and hence as above we find that the theorem holds for **H**.

The theorem is obviously true when the order h of **H** is unity and hence, by induction, it is true for any group.

In some cases the factors of a group are unique. This is so, for instance, when their orders are relatively prime, or more generally if the orders of their centrals are relatively prime. The following theorem, along with the corollaries to theorem 2, shows the connection between the factors when they are not unique.

THEOREM 4. *If* **A** *and* **B** *are any two factors of* **H**, **A** *being prime, then if* $\mathbf{A} \frown \mathbf{B} > I$, **B** *has a factor which is simply isomorphic with* **A**.

Since **B** is a factor, $\mathbf{AB} = \mathbf{C} \times \mathbf{B}$, and if $\mathbf{A} \frown \mathbf{B} > I$, then $c < a$. Hence **C** is not simply isomorphic with **A**, and therefore, by the previous theorem, **B** has a factor which is simply isomorphic with **A**.

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EXISTENCE OF THE GENERALIZED GREEN'S FUNCTION

By W. D. A. Westfall

In his second paper on the theory of linear integral equations,* Hilbert has established a relation between these equations and the boundary problems of ordinary, self-adjoined differential equations in the following manner:

If in

$$L(u) \equiv \frac{d}{ds}\left[p(s)\,\frac{du}{ds}\right] + q(s)\,u,$$

$p(s)$ is continuous, together with its first derivative, and $q(s)$ is continuous for $a \leqq s \leqq b$, then

$$(1) \qquad \int_a^b vL(u) - uL(v)\,ds = \Big[p(vu' - uv')\Big]_a^b,$$

provided u and v are continuous with their first and second derivatives. This is Green's formula for one dimension. A set of two linear, homogeneous boundary conditions that involve the vanishing of

$$\Big[p(vu' - uv')\Big]_a^b$$

for any two functions u and v, each satisfying these boundary conditions, is a set of Green's boundary conditions.

A continuous solution of

$$L(u) + \lambda u = 0,$$

with a continuous first derivative, and satisfying moreover a set of Green's boundary conditions, is called a characteristic solution. A continuous solution $G(s, t)$ of $L(u) = 0$, satisfying the boundary conditions, but having at $s = t$ the following discontinuity in its first derivative:

$$\operatorname*{Lim}_{\epsilon=0}\left[\frac{dG}{ds}\right]_{s=t+\epsilon} - \operatorname*{Lim}_{\epsilon=0}\left[\frac{dG}{ds}\right]_{s=t-\epsilon} = -\frac{1}{p(t)},$$

* *Göttinger Nachrichten*, 1904.

is called a Green's function. It follows from (1) that if u_i, u_j are characteristic solutions of

$$L(u_i) + \lambda_i u_i = 0,$$
$$L(u_j) + \lambda_j u_j = 0,$$

$\lambda_i \neq \lambda_j$, then

$$\int_a^b u_i(s)\, u_j(s)\, ds = 0. \tag{2}$$

With the Green's function $G(s, t)$ they satisfy the relation

$$u_i(s) = \lambda_i \int_a^b G(s, t)\, u_i(t) dt. \tag{3}$$

Moreover any function satisfying (3) with a Green's function $G(s, t)$ is such a characteristic solution of $L(u) + \lambda_i u = 0$. If the Green's function exists, the relation between integral equations and the boundary problems of differential equations is thus established.

If $p(s)$ is positive in $a \leqq s \leqq b$, it is known that either a Green's function or a characteristic solution of $L(u) = 0$ exists, but not both.* In case a characteristic solution exists, in which case there is no ordinary Green's function, Hilbert has defined a generalized Green's function $H(s, t)$, which replaces G in (3). If ω_1, ω_2 constitute a complete set of linearly independent solutions of $L(u) = 0$, normalized and orthogonalized, that is, such that

$$\int_a^b \omega_1 \omega_2 ds = 0, \quad \int_a^b \omega_1^2 ds = 1, \quad \int_a^b \omega_2^2 ds = 1,$$

then the generalized Green's function is the Green's solution $H(s, t)$ of

$$L(u) = \omega_1(s)\, \omega_1(t) + \omega_2(s)\, \omega_2(t),\dagger \tag{4}$$

for which

$$\int_a^b H(s, t)\, \omega_n(t)\, dt = 0, \qquad n = 1, 2. \tag{5}$$

There follow in the article referred to certain examples of this generalized Green's function, but no attempt is made to prove its existence in general. It is the purpose of this note to supply this proof.

* The writer's dissertation, Göttingen, 1905, p. 20.

† Hilbert considers only the case where only a single characteristic solution exists for $\lambda = 0$.

Let the two linearly independent solutions of $L(u) + \lambda u = 0$ be $z_1(\lambda, s)$ and $z_2(\lambda, s)$. These are analytic in λ.* If a characteristic solution exists for all values of λ it is given by

$$c_1 z_1(\lambda, s) + c_2 z_2(\lambda, s) = u(\lambda, s),$$

where c_1 and c_2 are so determined that the boundary conditions are satisfied. Hence such a solution is analytic in λ. From (2)

$$\int_a^b u(\lambda, s)\ u(\lambda + h, s)\ ds = 0,$$

or, by the theorem of the mean,

$$\int_a^b [u(\lambda, s)]^2\, ds + h \int_a^b u(\lambda, s) \frac{\partial u(\lambda + \theta h, s)}{\partial \lambda}\, ds = 0, \qquad 0 < \theta < 1,$$

which is impossible for h sufficiently small, unless $u(\lambda, s) = 0$.

There exists then for some value of λ, as $\lambda = \rho$, no characteristic solution, but a Green's function $G(\rho; s, t)$, since $L(u) + \rho u = 0$ is an equation of the same type as $L(u) = 0$, where $q + \rho$ replaces q. Let

$$E(\rho; s, t) = G(\rho; s, t) + \frac{1}{\rho} [\omega_1(s)\ \omega_1(t) + \omega_2(s)\ \omega_2(t)].$$

E is a Green's function of

$$L(u) + \rho u = \omega_1(s)\ \omega_1(t) + \omega_2(s)\ \omega_2(t),$$

such that

$$\int_a^b E(\rho; s, t)\ \omega_n(s)\, ds = 0, \qquad n = 1, 2. \tag{6}$$

From the theory of linear integral equations we know† that there is a continuous solution $K(s, t)$ of

$$E(\rho; s, t) = K(s, t) + \rho \int_a^b E(\rho; s, r)\ K(r, t)\, dr, \tag{7}$$

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* Picard: *Traité d'Analyse*, vol. 2, p. 92.

† I. Fredholm: *Acta Mathematica*, vol. 27 (1903).

or a not-identically vanishing solution of

$$\phi(s) = -\rho \int_a^b E(\rho; s, t)\, \phi(t)\, dt.$$

From (6) a solution of this latter equation is orthogonal to ω_1 and ω_2, that is

$$\int_a^b \omega_n(s)\, \phi(s)\, ds = 0, \qquad n = 1, 2. \tag{8}$$

It satisfies, moreover, all linear homogeneous boundary conditions satisfied by E in s, and is a solution of $L(u) = 0$, as may be seen by direct substitution. It must then be a characteristic solution of $L(u) = 0$. But equation (8) shows that such a function ϕ cannot exist, since ω_1 and ω_2 constitute a complete set of linearly independent characteristic solutions. There exists then a continuous solution $K(s, t)$ of (7). It is seen immediately that this is the generalized Green's function sought.

It has been assumed in the above proof that two linearly independent characteristic solutions of $L(u) = 0$ exist. Neglecting the case where only one exists has affected the proof only in simplifying the notation. The following theorem has been shown:

If $p(s)$ is continuous together with its first derivative, and positive, and if $q(s)$ is continuous in a given closed interval, there exists for each set of Green's boundary conditions for this interval either a Green's solution or a generalized Green's solution of $L(u) = 0$.

This proof, with a mere change in notation, holds for self-adjoined linear differential equations of the $2n$th order.

COLUMBIA, MO.,
FEBRUARY, 1909.

THE GAMBLER'S RUIN

By J. L. Coolidge

The various problems which are connected with games of chance have interested mathematicians ever since the time of Pascal. In particular, the probability that when two men are playing together at a certain game, the one will end by ruining the other, and the probable number of turns before one player's ruin, has been studied by James Bernouilli, DeMoivre, Lagrange, Laplace, and others.* In all of these works, however, it is assumed that the amount of the stakes is the same at each turn, and the problem of showing that, regardless of the amount which is staked, the probability of ruin for the one or the other player, lies within certain definite limits, has not, so far as I have been able to discover, ever been undertaken. In general the mathematician assumes, without any proof, that whoever indulges in a game favorable to his adversary has a very strong chance of being ruined, regardless of what system of play he may affect. The player, on the other hand, ascribing to the goddess of fortune a desire which she does not possess, to atone for past errors, works out a system of play in which he confides. It is the object of the present paper to show that no such system can have more than an easily calculable chance of success.

1. Unlimited Stakes. Let us imagine that two persons called respectively the "player" and the "banker" are playing at a game of chance. The effective difference between the two is that the player has the right to settle the amount of the stake each time; and also to leave off when he chooses. Suppose, first, that the player has a capital which, compared with the banker's, may be roughly called indefinitely great. Such is, in fact, approximately the case when the banker conducts a public gaming establishment ready to meet all comers. The probability that the player will be ruined is, here, negligible. On the other hand if the player begin by staking the amount of the banker's fortune and doubles his stake every time he loses, the probability that he will end by ruining the banker, is very large indeed. For that reason public gaming establishments habitually set an upper limit to the amount which

* See Todhunter's *History of the Theory of Probability*, London, 1865.

may be staked. Let us, secondly, suppose that whereas the player has only a fortune of m the banker's fortune is quasi-infinite in comparison therewith. Let us say that a first game is finished when either the player is ruined, or he has doubled his original fortune. As in this game his stake is surely less than $2m$ we are under the second case of limited stakes, to be treated presently, and the chance that the player will escape ruin is not over one-half; unless the game is favorable to him, as is not the case in public gaming. If the player win the first game, let him start another which shall terminate when the one side or the other has won $2m$. Once more his chance of escaping ruin is $\frac{1}{2}$. Keeping on thus we see that if the player undertake to play an indefinite number of games, the chance that he will never be ruined is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \cdots$; indefinitely small.

If the player and banker have each a finite sum, the maximum stake is surely less than their combined fortunes, and we are under our second case.

2. Limited Stakes. Let us assume that the player has a fortune m and that he undertakes to play until either he losses the whole of this, or wins the sum n from the banker. This is the usual state of affairs in public gambling, although some players being with a very indefinite notion of the value of n, which has the effect of making n very large, thereby greatly increasing the probability that the player will be ruined. When the player has won the sum n we shall, for convenience, say that the bank has been ruined, though the player of ordinary means who should really set n at the amount of the bank's funds, would do a very foolish thing. For the sake of simplicity we shall assume that if the player is so nearly ruined that he cannot afford to stake as much as he wishes, then he stakes all that he has, whereas if his winnings are so near to n that the residue is less than the banker should put up in order to cover the player's usual stake, then also we shall assume that the player reduces his stake accordingly. As a matter of fact it does not make any great difference how we treat these extreme circumstances; unless $m + n$ is a comparatively small multiple of the amount of the stake, or unless the game is grossly unfavorable to one side; when either of these cases arises, the ruin of the less favored party is almost certain.

What do we mean by the game's being favorable or unfavorable to one side?

If a man have mutually exclusive chances $p_1\, p_2 \cdots p_r$ where $\sum_{i}^{1=r} p_i = 1$

of obtaining the positive or negative values $a_1\ a_2 \cdots a_r$ respectively, then the expression

$$\sum_{i=1}^{i=r} p_i a_i$$

is defined as his *expectation*. An individual play, or a whole game, is said to be *fair* when the expectation of each person before entering is zero. Thus, let the player stake a with the probability p of winning, while the banker, whose favorable chance is $q = 1 - p$, replies with a stake b. The player's expectation is $pb - qa$, while that of the banker is $qa - pb$. The criterion for a fair play is $pb - qa = 0$,

or
$$\frac{a}{p} = \frac{b}{q}. \tag{1}$$

In a fair play between two persons, the stake of each is proportional to the probability that he will win.

This rule is easily extended to any number of persons.

Suppose, next, that we have a succession of two fair plays, the player's and banker's chances and stakes being respectively, a_1, p_1; b_1, q_1 in the first, a_2, p_2; b_2, q_2 in the second. The player's expectation from the game composed of these two is

$$p_1 p_2(b_1 + b_2) + p_1 q_2(b_1 - a_2) + p_2 q_1(b_2 - a_1) - q_1 q_2(a_1 + a_2)$$
$$= (p_1 + q_1)(p_2 b_2 - q_2 a_2) + (p_2 + q_2)(p_1 b_1 - q_1 a_1) = 0.$$

This reasoning may be continued indefinitely, whence

Any succession of fair plays will constitute a fair game.

The commonest games are those where p and q retain constant values; let us show that if such a fair game be continued indefinitely the ruin of either player or banker is certain. The player's fortune is set at m, the banker's at n. We first suppose that the stakes are constant, a and b. If v turns be played, the probability that the player shall win not more than $vp + k$ nor less than $vp - k$ is, by Bernouilli's theorem,

$$\frac{2}{\sqrt{\pi}} \int_0^{\frac{k}{\sqrt{2vpq}}} e^{-t^2}\, dt$$

and for any constant value of k, this will approach zero as a limit as v increases indefinitely. Replacing k by κ, which is supposed to be either positive or negative, if the player win $vp + \kappa$ turns his winnings will be

$$(vp + \kappa)\, b - (vq - \kappa)\, a = \kappa(a + b).$$

Take κ numerically larger than the larger of the two quantities $m/(a + b)$, $n/(a + b)$, and allow the play to continue indefinitely; the ruin of the one or other party is seen to be certain.

The case where the player may alter his stake at will requires more delicate handling. Let us suppose that in v turns the player makes the stake a_1, r_1 times, in general the stake, a_i, r_i times; the banker's corresponding stakes being $b_1, b_2, \cdots b_r$.

We have
$$\sum_{i=1}^{i=r} r_i = v.$$

The player's chance of winning a particular play is independent of the amount staked, so that of the $vp + \kappa$ plays which he wins, the most probable number to be associated with the stake, a_i, will be $(vp + \kappa)\,\frac{r_i}{v}$. Let us assume, therefore that, of the a_i play he wins $r_i p + \frac{\kappa r_i}{v} + \epsilon_i$. His total winnings will be

$$\sum_{i=1}^{i=r}\left[\left(p + \frac{\kappa}{v}\right) r_i + \epsilon_i\right] b_i - \sum_{i=1}^{i=r}\left[\left(q - \frac{\kappa}{v}\right) r_i - \epsilon_i\right] a_i$$
$$= \sum_{i=1}^{i=r}\left(\frac{\kappa}{v}\, r_i + \epsilon_i\right)\left(a_i + b_i\right).$$

Let us modify this slightly. Since $vp + \kappa$ turns are won,

$$\sum_{i=1}^{i=r} \epsilon_i = 0.$$

If, then a_1 be the smallest stake, we may write the player's winnings as

$$\frac{\kappa}{v}\sum_{i=1}^{i=r} r_i(a_i + b_1) + \sum_{i=2}^{i=r} \epsilon_i[(a_i - a_1) + (b_i - b_1)].$$

The quantity κ is, by Bernouilli's theorem, equally likely to be positive or negative. If, after v_1 turns, it be found to be positive, after v_2 turns more it will have a greater chance of being positive than negative, but this disproportion approaches zero as a limit, as v_2 increases indefinitely, and may be disregarded when v_2 is very large. The quantities $\epsilon_2 \cdots \epsilon_r$ may be positive or negative, and may all have like signs, or opposite signs in varying proportions. Let us assume, hence, that P represents the probability that after v_j turns, the sign of ϵ_i shall be that of κ; the probability that in t series of $v_1, v_2 \cdots v_t$ turns, each, the ϵ_i's shall never after v_j turns, all have the same sign as κ is

$$\prod_{j=1}^{j=t}\left(1 - \prod_{i=2}^{i=r} P_i^{(v_j)}\right),$$

and this quantity approaches zero as a limit as t increases indefinitely. But when all the ϵ's but ϵ_1 have the same sign as κ, the ruin of the one party or the other is even more certain than in the case of constant stakes.

The ruin of one party or the other being thus assured, we may easily calculate the probability favorable to each. Let P be the probability that the player shall ruin the banker, and $Q = 1 - P$ the probability that the banker shall ruin the player. As the game is fair, the expectation of each is zero, hence

$$Pn - Qm = 0, \qquad P = \frac{m}{m+n}, \qquad Q = \frac{n}{m+n}. \tag{2}$$

It should be noticed that these values are independent of the amounts staked.

We must now pass to the case of a game which is unfavorable to the player. Here

$$pb - qa < 0. \tag{3}$$

a and b shall be constant stakes, except in the extreme cases alluded to, where the one side or the other could not afford such a stake. Let us write the equation

$$f(x) \equiv px^{a+b} - x^a + q = 0,$$

$$f'(x) \equiv p(a+b)x^{a+b-1} - ax^{a-1}.$$

The roots of $f'(x)$ are

$$x = 0, \qquad x = \sqrt[b]{\frac{a}{p(a+b)}},$$

so that there is but one positive root. $f(x)$ has a root 1. Moreover

$$f'(1) = p(a+b) - a = pb - qa < 0,$$
$$f(\infty) = +\infty,$$

so that $f(x)$ has just one positive root, greater than unity. Call this α. Then

$$pa^{a+b} - a^a + q = 0, \qquad a > 1. \tag{4}$$

We now make use of an ingenious device due to DeMoivre,* which consists in assuming that the player's fortune is composed of m counters whose values are a, a^2, $\cdots a^m$ while the banker has n counters a^{m+1}, a^{m+2}, $\cdots a^{m+n}$. The player shall stake his a highest counters each time, and the banker his b lowest, so that when the lowest counter staked is a^x the player's expectation from the play is

$$p(a^{x+a} + a^{x+a+1} \cdots a^{x+a+b-1}) - q(a^x + a^{x+1} \cdots + a^{x+a-1})$$
$$= \frac{a^x}{a-1}[p(a^{a+b} - a^a) - q(a^a - 1)]$$
$$= \frac{a^x}{a-1}[pa^{a+b} - a^a + q] = 0.$$

The game has thus been transformed into a fair one, the ruin of the one or the other is certain, and the probability favorable to each is equal to his fortune divided by the sum of the two fortunes. The player's chance is, thus,

$$\frac{\sum_{i=1}^{i=m} a^i}{\sum_{i=1}^{i=m+n} a^i} = \frac{a^m - 1}{a^{m+n} - 1}. \tag{5}$$

The banker's will be

$$\frac{a^{m+n} - a^m}{a^{m+n} - 1}. \tag{6}$$

It seems to have escaped the notice of previous writers upon this subject that unless a and b are equal, and divide evenly into m and n, the values of a

* *The Doctrine of Chances*, London, 1756, p. 52.

or $m + n$ must be susceptible of modification in this formula, in order that it cover the case when one side or the other can not put up the requisite stake. This fact is theoretically important, but practically insignificant, as we have already pointed out. When the player's fortune has sunk to $\epsilon < a$, the chance favorable to him is certainly less than $\epsilon/(m + n)$, a trivial fraction, when n is large compared with a. On the other hand, when the banker's fortune is reduced to $\eta < b$, if the game be nearly fair, his chance is but little over $\eta/(m + n)$.

It is now time to take up the elusive case where the player may alter his stake at will. Assuming that a is the largest allowable stake, let us imagine that the player leads off with a stake a' less than a, but constantly plays a thereafter. Has he improved or injured his chance? The latter is, in the present case, made up of two parts, according as he wins or loses the first play, so that it has the value

$$p\,\frac{a^{m+b'}-1}{a^{m+n}-1}+q\,\frac{a^{m-a'}-1}{a^{m+n}-1}. \tag{7}$$

Subtracting this expression from (5) we get

$$\frac{a^m-pa^{m+b'}-qa^{m-a'}}{a^{m+n}-1}=\frac{a^m}{a^{a'}(a^{m+n}-1)}(-pa^{a'+b'}+a^{a'}-q).$$

If we write $a' = a\rho$, $b' = \rho b$, $\rho < 1$ we see $\alpha' = a_\rho^{\frac{1}{\rho}}$ is a root of the equation

$$px^{a'+b'}-x^{a'}+q=0,$$

and

$$1<\alpha<a_\rho^{\frac{1}{\rho}},$$

$$p\alpha^{a'+b'}-\alpha^{a}+q<0.$$

The difference between (7) and (5) is positive, or the probability favorable to the player has been lessened.

There is just one troublesome case where this reasoning is inapplicable, namely, where, after losing a' the player would have to reduce his stake, thus altering a in the second term of (7).

The player's stake, in case he lose the first play, shall now be

$$m-a'=\frac{a}{\sigma}\qquad\sigma>1$$

We must consequently increase a to a^σ.

The player's chance thus becomes

$$p\,\frac{a^{m+b'}-1}{a^{m+n}-1}+q\,\frac{a^{\sigma(m-a')}-1}{a^{\sigma(m+n)}-1}. \tag{8}$$

It is necessary to compare the second term of (8) with that of (7). To do so let us note the alteration in the former as σ increases from 1. Expanding by Maclaurin's theorem we have

$$\frac{a^{\sigma(m-a')}-1}{a^{\sigma(m+n)}-1}=\frac{\Big[(m-a')\log a\Big]\sigma+\frac{1}{2!}\Big[(m-a')\log a\Big]^2\sigma^2+\cdots}{\Big[(m+n)\log a\Big]\sigma+\frac{1}{2!}\Big[(m+n)\log a\Big]^2\sigma^2\cdots}.$$

Let us write this

$$\frac{f(\sigma)}{\phi(\sigma)}=\frac{c\sigma+\frac{c^2}{2!}\sigma^2+\frac{c^3}{3!}\sigma^3+\cdots}{C\sigma+\frac{C^2}{2!}\sigma^2+\frac{C^3}{3!}\sigma^3+\cdots}, \qquad c<C$$

$$\frac{d}{d\sigma}\left[\frac{f(\sigma)}{\phi(\sigma)}\right]=\frac{\frac{cC}{2}\Big(c-C\Big)\sigma^2+\frac{cC}{3}\Big(c^2-C^2\Big)\sigma^3+\cdots}{\Big(C\sigma+\frac{C^2}{2!}\sigma^2+\cdots\Big)^2}.$$

This expression is negative for $\sigma>1$, hence (8) is less than (7), and, a fortiori less than (5).

We may pursue this line of reasoning further. As it was a mistake for the player to replace his first stake by another smaller stake, so it will be a further error for him to replace a in his second stake by a smaller value, and so on. We have already seen that any series of stakes whatsoever is sure to end in the ruin of the one party or the other, and we now see that no other series gives the player such a good chance as he gets by playing the maximum each time. We thus reach our fundamental theorem.

The player's best chance of winning a certain sum at a disadvantageous game, is to stake the sum which will bring him that return in one play, or, if that be not allowed, to make always the largest stake which the banker will accept.

The principal deduction from this is of a moral nature. The average gambler will say "The player who stakes his whole fortune on a single play is a fool, and the science of mathematics can not prove him to be otherwise." The reply is obvious: "The science of mathematics never attempts the impossible, it merely shows that other players are greater fools."

Let us make some simple deductions from our formula (5). The player's chance is always less than $m/(m+n)$. A better approximation, when m and n are both small is

$$\frac{m + \frac{1}{2}\, m(m-1)(a-1)}{(m+n) + \frac{1}{2}\,(m+n)(m+n-1)(a-1)}.$$

When m and n are equal the player's chance is

$$\frac{1}{a^m + 1}. \tag{9}$$

If the player should become frightened, and reduce his stake to $\frac{a}{\sigma}$, his chance would be reduced to

$$\frac{1}{a^{\sigma m} + 1}.$$

When a and b are equal

$$a = \frac{q}{p}. \tag{10}$$

When m is decidedly large, the probability that the banker will be ruined is close to

$$\frac{1}{a^n}. \tag{11}$$

3. The Game of Roulette. As an application of the foregoing principles, I am going to consider the game of roulette, as played at Monte Carlo. This game has two advantages, the honesty of the banker is above suspicion, and the player's disadvantage is less than in any other form of gambling with which I am familiar. The following description is taken from Sir Hiram Maxim.*

"The roulette consists of a large circular basin, about two feet in diameter, with the outer rim turned inward. The bottom of the basin, which forms the wheel,

* *Monte Carlo, Facts and Fancies*, London, 1906, pp. 257 ff.

is of metal quite separate from the rim or sides, and is nicely balanced and mounted on a fine pivot so that when set in motion, it will spin for a considerable time. The outer edge of the wheel is accurately divided into thirty-seven sections of pockets, eighteen of which are painted red and eighteen black. One is called zero, and is neutral in color. The pockets are numbered from one to thirty-six." The wheel is set in motion, and a small ball started rolling around the edge in the opposite direction. The game consists in betting on the number or color of the division into which the ball will eventually roll. There are fourteen different methods of staking. The simplest are, red or black, even or odd, above or below eighteen. In each of these cases the player and banker stake equal sums. If a player stake upon a number, the banker puts up thirty-five times the amount. The upper limit for a simple stake, say red or black, is twelve hundred dollars; a player may not stake more than thirty-six dollars on a number. When the ball falls into zero, the player on a simple chance may either forfeit one half of his stake or leave the whole "in prison" till the next turn. If he be fortunate at this turn he saves his stake but gets nothing further, if he lose, the stake is gone forever. On the other hand all who have staked on particular numbers or combinations thereof lose their stakes at once on the arrival of zero.

What are the probabilities of success for a player staking on a simple chance as red or black? We may imagine that he plays twelve hundred dollars each time, and this sum we shall call one, so that m and n represent the player's and the banker's fortunes respectively, expressed as multiples of twelve hundred. In seventy-four turns the player may expect on an average, thirty-six reds, thirty-six blacks, one zero followed by a loss, and one zero followed by a gain, which leaves matters just as before and may be counted out. We have then

$$p = \frac{36}{73}, \quad q = \frac{37}{73},$$

$$a = b = 1, \quad a = \frac{37}{36}.$$

The player's probability of winning is

$$\frac{\left(\frac{37}{36}\right)^m - 1}{\left(\frac{37}{36}\right)^{m+n} - 1}.$$

When m and n are equal this becomes

$$\frac{1}{\left(\frac{37}{36}\right)^m + 1}.$$

Let us next suppose that the player stakes upon a number. The amount which may be staked is, here, about one thirty-third of what it was before, so that the game is the same as if the stake remained constant, and the fortunes of both banker and player were increased thirty-three fold. The probability for the player will be the same as before if a be equal to the thirty-third root of $\left(\frac{37}{36}\right)$, and will be less if a be larger. Now the thirty-third root of $\frac{37}{36}$ is 1.00085.

To find a, we have $p = \frac{1}{37}$, $q = \frac{36}{37}$, $a = 1$, $b = 35$, $a = 1 + \epsilon$,

$$(1 + \epsilon)^{35} - 37(1 + \epsilon) + 36 = 0,$$

$$(1 + 36\,\epsilon + 630\,\epsilon^2) - 37 - 37\,\epsilon + 36 = 0,$$

$$1 + \epsilon = \frac{631}{630} = 1.0015.$$

This method is, therefore, somewhat less favorable to the player than the other.

A third simple game is played by staking on all the numbers but one. The maximum allowable here is thirty-five times thirty-six dollars, not far from twelve hundred, so that we may call the stake one as before. Each time the player wins at this game he sacrifices thirty-four thirty-fifths of his stake, so that b is one thirty-fifth. We shall further have

$$p = \frac{35}{37}, \quad q = \frac{2}{37},$$

$$qa - pb = \frac{1}{37},$$

whereas in the simple game

$$qa - pb = \frac{1}{73},$$

which shows that staking on thirty-five numbers would be an unwise method of play.

When n is large, the chance of ruin for the bank is

$$\frac{1}{a^n}.$$

We have no accurate data as to the amount of n, but apparently as the bank's earnings in a year are between five and six million dollars* it is not unreasonable to suppose that the sum of one million two hundred thousand dollars might be produced in a very short time. The chance of ruin on red or black is therefore not above

$$\left(\frac{36}{37}\right)^{1000.} = \frac{1}{792870000000}.$$

I conclude with two more quotations from Maxim's book. The first is from a letter from Baron Czyllak, quoted on p. 172.

"The calculation of probabilities by means of which inventors of systems seek to master roulette, and also the theory of roulette, have for their author the same savant that is to say, the celebrated Pascal." So does posterity treat the great theologian whose treatise on cycloids is called "Histoire de la roulette!"

The next quotation is from Maxim himself, on p. 190, and gives the best explanation which I have seen for the fact the people continue to gamble:

"Je me rends parfaitement compte du désagreable effet que produit sur la majorité de l'humanité, tout ce qui se rapporte, même au plus faible dégré, à des calculs ou raisonnements mathématiques."

CAMBRIDGE, MASS.,
MARCH, 1909.

* *Maxim*, loc. cit. p. 9.

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